Question

(*) A tank contains a well-stirred solution of 5 kg salt and 500 L of water. Starting at $t = 0$, fresh water is poured into the well-stirred solution at a rate of 4L/min, and the mixture leaves at the same rate.

1. What is the differential equation governing the amount $x(t)$ of salt in the tank at time t?
2. How long will it take for the concentration of salt to reach a level of 0.1%?
3. The next day the procedure is repeated but the exit pipe has become partially blocked so that the mixture only leaves at a rate of 2 L/min. The capacity of the tank is 1000 L, what is the concentration of the mixture when it first overflows?

Answer

a) $V(t)$ =volume of water in tank in Litres. $\quad V(0) = 500$

$S(t)$ =mass of solution in tank in kg. $\quad S(0) = 5$

Now balance the water:

$$\begin{array}{ccccc} \begin{array}{c} \textit{rate of change} \\ \textit{of volume} \end{array} & = & \textit{rate of water in} & - & \textit{rate of water out} \\ \frac{dV}{dt} & = & 4 & - & 4 \end{array}$$

$$\Rightarrow \frac{dV}{dt} = 0 \Rightarrow V(t) = 500$$

Now balance the salt:

$$\begin{array}{ccccc} \begin{array}{c} \textit{rate of change} \\ \textit{of salt} \end{array} & = & \textit{rate of salt in} & - & \textit{rate of salt out} \\ \frac{dS}{dt} & = & 0 & - & 4\left(\frac{S}{V}\right) \end{array}$$

(no salt is coming in and the concentration in the tank is $\dfrac{S}{V}$).

$$\frac{dS}{dt} = -\frac{4S}{500} \Rightarrow S = 5e^{\left(-\frac{4t}{500}\right)}$$

b) The concentration of salt gets to 0.1% of initial value when

$S = 0.005$ i.e. $0.005 = 5e^{-\frac{4t}{500}} \Rightarrow t = 863\text{minutes} \approx 14\text{hours}$

c) Water balance is:

$$\frac{dV}{dt} = 4 - 2 \Rightarrow \frac{dV}{dt} = 2 \Rightarrow V(t) = 500 + 2t$$

hence it overflows at $t = 250 mins$

Salt balance is:

$\frac{dS}{dt} = -\frac{2S}{V} \Rightarrow \frac{dS}{dt} = -\frac{2S}{500 + 2t}$ solve by separation of variables.

$$\int \frac{1}{S} dS = \int \frac{-2}{500 + 2t} dt \Rightarrow \ln S = -\ln(500 + 2t) + A$$

$S(0) = 5$ then gives $\ln \mathrm{S} = -\ln(500 + 2\mathrm{t}) + \ln(500) + \ln(5)$

$$S(t) = 5\left(\frac{500}{500 + 2t}\right)$$

concentration $= \frac{S(t)}{V(t)} = 5\left(\frac{500}{500 + 2t}\right)\left(\frac{1}{500 + 2t}\right)$

and at t=250 mins (at overflow) $\frac{S}{V} = 2.5 * 10^{-3} kg/L$