FUNCTIONAL ANALYSIS
METRIC SPACES

A metric space is a space in which is defined a distance function p with the following properties:

1. $p(x,y) \geq 0$ with equality $\Leftrightarrow x = y$.
2. $p(x,y) = p(y,x)$.
3. $p(x,y) \leq p(x,z) + p(y,z)$.

A sequence $\{x_n\}$ is a cauchy sequence if, given $\varepsilon > 0 \exists N$ such that $p(x_n, x_m) < \varepsilon$ whenever $n, m > N$.

Definition A complete metric space is a metric space in which every Cauchy sequence converges.

Contraction Mapping Theorem Let X be a complete metric space and T a continuous mapping of X into X such that $p(Tx, Ty) \leq \theta(p(x,y))$ where $0 < \theta < 1$. Then T has a unique fixed point.

Proof Let x be an arbitrary point in X. Let $x_0 = x,\ x_1 = T(x),\ x_2 = T(T(x))\ldots$.

The sequence $\{x_n\}$ is a cauchy sequence for $p(x)n+1, x_n) \leq \theta^n p(x_0, x_1)$ and so

$$\begin{aligned} p(x_{n+p}, x_n) &\leq p(x_{n+1}, x_n) + \ldots + p(x_{n+p}, x_{n+p-1}) \\ &\leq (\theta^n + \theta^{n+1} + \ldots + \theta^{n+p-1}) p(x_0, x_1) \end{aligned}$$

and therefore the coefficient of $p(x_0, x_1)$ can be made arbitrarily small provided only that n is large.

Therefore there is a point $\xi \in X$ such that $x_n \to \xi$ as $n \to \infty$ $x_0 x_1 x_2 \ldots \to \xi$.

Therefore $Tx_0 Tx_1 \ldots \to T\xi$ by continuity of T i.e. $x_1 x_2 \ldots \to T\xi$ therefore $T\xi = \xi$.

Suppose $\eta = T\eta,\ \xi = T\xi$.

$p(\xi, \eta) = p(T\xi T\eta) \leq \theta p(\xi, \eta)$ $\theta < 1$therefore $p(\xi, \eta) = 0$ therefore $\xi = \eta$.

Exercise
$$4\frac{dx}{dt} + \sin x + \int_{\frac{t}{2}}^{t} 1 + x^2(s)\sin s\, ds = 0$$

prove that there is a unique function $x(t)$ defined and continuous for $0 \leq t \leq 1$ which is a solution of this equation.

Definition A set H is nowhere dense if $(\overline{H})^0 = \emptyset$.

Baire's Theorem Let X be a complete metric space and $\{H_n\}$ a sequence of subsets of X such that $\cup H_n = X$. Then it is impossible for every one of the H_n to be nowhere dense.

Proof Suppose the H_n are all nowhere dense. Then the complement of $\overline{H}_1$ contains some non-empty sphere S_1 where $r(S_1) < 1$. Since S_1 is not contained in $\overline{H}_2$ we can find a non-empty closed sphere S_2 contained in the complement of $\overline{H}_2$ and in S_1.

We may suppose $r(S_2) < \frac{1}{2}$.

Proceeding in this way we can find a sequence of non-empty closed spheres $S_1 \supset S_2 \supset \dots$ with $\overline{H}_n \cap S_n = \emptyset$ and $r(S_n) < \frac{1}{n}$.

For each n suppose $x_n \in S_n$. Then $\{x_n\}$ is a Cauchy sequence therefore $x_n \to x$ as $n \to \infty$ therefore $x \in \cap_{n=1}^{\infty} S_n$ therefore $x \in \cup_{n=1}^{\infty} H_n$ which is a contradiction.

Baire's theorem is also true for locally compact spaces.

Definition A set of the first category is a set which is the join of an enumerable number of nowhere dense sets.

A set of second category is one which is not of the first category.

Example The rational numbers cannot be expressed as an intersection of open sets.

Proof The real line is a complete metric space and so by Baire's Theorem is of the second category. The rationals, being a countable set, are a set of the first category. Suppose that $\cap_{i=1}^{\infty} G_i = \mathcal{Q}$ (the rationals).

Then F_1' is closed and $\cup_{i=}^{\infty} G_i' = \nabla - \mathcal{Q}$.

$)\overline{G}_2')^0 = (G_i')^0 = \emptyset$, for otherwise G_2' would contain an interval, and so would contain points of $\mathcal{Q}$. But this tells us that $\mathcal{R} - \mathcal{Q}$ is of the first category which is a contradiction.

Example The set of continuous functions which are differentiable at even one point is a set of the fist category.

Proof Let f be defined and continuous on $[0\ 1]$. Define $p(f,g) = \sup_{x\in[0\ 1]}\{|f(x)-g(x)|\}$. Then this is a complete metric space, denoted by $L^p[0\ 1]$.

Consider the set of functions which are differentiable at 0 say, i.e. $\frac{f(x)-f(0)}{x} \to$ limit as $n \to \infty$.

Define $H_{m,n} = \left\{f : \left|\frac{f(x)-f(0)}{x}\right| \leq n \text{ whenever } x < \frac{1}{m}\right\}$.

Each $H_{m,n}$ is closed and is nowhere dense, and $\cup_{m,n} H_{m,n}$ contains the set of all functions differentiable at the origin. Hence the set of functions differentiable at the origin is in the first category.

The argument can be extended as follows. Let $U_1 u_2 \ldots$ be an enumerable basis for the open sets e.g. the rational spheres with rational centres, and define

$$H_{m,n} = \left\{f : \left|\frac{f(x)-f(y)}{x-y}\right| \leq n \text{ whenever } x, y \in U_m\right\}$$

Zorn's Lemma (A form of the axioms of choice) Any partially ordered set in which every simply ordered subset has a maximum element, has a maximum element.

A Hamel basis for the real numbers is a set B such that

(a) Any subset of B is rationally independent.

(b) Any real number is a t.c. of a finite number of the B.

Consider all subsets Y of R_1 with the property that $y_1 \ldots y_n \in Y \Rightarrow r_1 y_1 + \ldots + r_n y_n \neq 0$ unless $r_1 = \ldots = r_n = 0$, r rational.

We order the sets Y by inclusion.

Let $\{Y_\alpha\}$ be a simply ordered class of these sets.

If $Y = \cup_\alpha$ then Y has the required independence property, for if $y_{\alpha_i} \in Y$ $(i = 1, \ldots, n)$ then $y_{\alpha_i} \in Y_i$ $i = 1, \ldots, n$ and $\exists \max_{i=1,\ldots,n} Y_i$ for $\{Y_\alpha\}$ is simply ordered so the y_{α_i} are rationally independent.

Hence y is a maximum element, therefore by Zorn's Lemma $\exists$ a set B which is a maximal element.

Suppose r cannot be expressed as a r.c. of elements in B. Then we add x to B to get a greater set with the same independence property therefore as B is maximal it follows that every real x can be expressed as a r.c. of elements in B.

The same method of proof also shows the existence of a basis for any vector space whatsoever.