FUNCTIONAL ANALYSIS
HAHN-BANACH THEOREM

If $M$ is a linear subspace of a normal linear space $X$ and if $F$ is a bounded linear functional on $M$ then $F$ can be extended to $M + [x_0]$ without changing its norm.

**Proof** We first suppose that $X$ is a vector space over the real numbers. We may suppose without loss of generality $\|F\| = 1$. We may define $F(x_0) = \alpha$ and $F(m + \lambda x_0) = F(m) + \lambda\alpha \; m \in M$.

We must have

$$|F(m) + \lambda\alpha| \leq \|m + \lambda x_0\|$$

$$\text{i.e. } |F(m) + \alpha| \leq \|m + x_0\| (m \text{ arbitrary} \Rightarrow \tfrac{m}{\lambda} \text{ arbitrary}$$

If $m_1 \; m_2 \in M$

$$\begin{aligned} F(m_1) + \alpha &\leq \|m_1 + x_0\| \\ F(m_2) + \alpha &\geq -\|m_2 + x_0\| \\ -F(m_2) - \|m_2 - x_0\| &\leq \alpha \leq -F(m_1) + \|m_1 + x_0\| - \;\; (I) \end{aligned}$$

$$\begin{aligned} F(m_1) - F(m_2) &= F(m_1 - m_2) \leq \|m_1 - m_2\| \\ &\leq \|m_1 + x_0\| + \|m_2 + x_0\| \end{aligned}$$

therefore $\exists\alpha$ satisfying (I).

Now if $X = X(\mathcal{C})$

$$\begin{aligned} F(x) &= G(g) + iH(x) \\ iF(x) &= F(ix) = iG(x) - H(x) \text{ therefore } H(x) = -G(ix) \\ &= G(ix) + iH(x) \text{ therefore} \\ F(x) &= G(x) - iG(ix) \end{aligned}$$

and $G$ can be extended by the first part.

By Zorn's lemma there will be a maximal subspace $N$ to which $M$ can be extended and $N = X$, applying the theorem.

We can embed $X$ in $X^{**}$ as follows:

Let $x \in X$ and define $f(\lambda x) = \lambda \|x\|$. Then $\|f\| = 1$.

$f$ can be extended to the whole space without changing its norm.

$$\begin{aligned} |\tilde{x}(f)| &= |f(x)| = \|x\| \\ \text{therefore } \|\tilde{x}\| \geq \|x\| & \end{aligned}$$

Also $|\overline{x}(f)| = |f(x)| \leq \|f\|\|x\|$ therefore $\|\overline{x}\| \leq \|x\|$.

**Adjoint of an operator** Let $T$ be a continuous linear transformation from $X \to Y$.

The adjoint $T^*$ of $T$ is a linear transformation from $Y^*$ to $X^*$ defined as follows:

Let $f \in Y^*$.

We define $T^*(f) \in X^*$ by $(T * f)x = f(TX)$

$$\begin{aligned} \|T^*(f)\| &= \sup_{\|x\|=1} |fT(x)| \\ &\leq \|f\| \sup_{\|x\|=1} \|Tx\| \\ &\leq \|F\|\|T\| \end{aligned}$$

Therefore $T^*$ is continuous and $\|T^*\| \leq \|T\|$.

Now $T^{**}$ maps $X^{**}$ to $Y^{**}$.

If $X$ is regarded as a subspace of $X^{**}$ then $T^{**}$ is an extension of $T$ therefore $\|T\| \leq \|T^{**}\| \leq \|T^*\|$ therefore $\|T^*\| = \|T\|$.

**Weak topology** Let $X$ be a normed vector space and let $X^*$ be the dual of $X$. We define a topology on $X$, called the weak topology, by taking the sets

$$V(x)_{f_1 \ldots f_n \varepsilon} = \{y \in X : |f_i(x) - f_i(y)| < \varepsilon \ i = 1, \ldots, n\}$$

as a basis of neighbourhoods of the point $x$, where $f_1 \ldots f_n$ are any functionals in $X^*$ and $\varepsilon$ is any positive number.

As all the $f$ are continuous this set will be open in the original topology and so this topology is weaker than the original one.

**Example** Let $\xi = (x_n) \in \ell^2$

Let $f = (y_n)$

$$f(\xi) = \sum x_n y_n = (\xi, \eta).$$

$\xi_\alpha \to 0$ in the weak topology $\Leftrightarrow (|xi_\alpha, \eta) \to 0$.

let $\varepsilon_1 = (1, 0, 0, \ldots)$ $\varepsilon_2 = (0, 1, 0, \ldots)$ etc.

$\|\varepsilon_m - \varepsilon_n\| = \sqrt{2}$ $m \neq n$. But for the weak topology this sequence converges to zero as $\varepsilon_n$ $\eta) = y_n \to 0$ as $\sum |y_n|^2 < \infty$.

**Example** Consider the space of all real valued functions defined on [0 1].

Consider the topology given by $f_\alpha \to f \Leftrightarrow f_\alpha(x) \to f(x)$ for each $x$ in [0 1].

Basic neighbourhoods:

Given $x_1 \ldots x_n$ and $\varepsilon > 0$

$$N = \{g : |f(x_i) - g(x_i)| < \varepsilon \ i = 1 \ldots n$$

Let $C$ be the subspace of continuous functions. Let $d(x) = \begin{cases} 1 & x \text{ irrational} \\ 0 & x \text{ rational} \end{cases}$

$d \in \overline{C}$ for this topology, for given $x_1 \ldots x_n$ we can find $f \in C$ such that

$$f(x_i) = d(x_i) \ i = 1, \ldots, n$$

and so $f \in N(d)$.

But no sequence of continuous functions converges to $d$ in this topology for, given $\{f_n\} \in C$ and $f_n(x) \to d(x)$ at every $x$.

Let $H_n = \cap_{r \geq n}\{x : f_r(x) \geq \frac{1}{2}\}$. $H_n$ is closed. $H_n$ contains no rational and so is nowhere dense, therefore $\cup_{n=1}^{\infty} H_n$ is of the first category. But $\cup_{n=1}^{\infty} H_n$= irrationals - of second category.

**Theorem** Suppose $X$ is a Banach space, than the unit sphere of $X^*$ is compact in the weak $*$ topology.

**Proof** For each $x \in X$ define

$$C_x = \{z : |z| \leq \|x\|\}$$

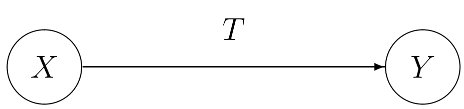
$c_x$ is a compact set therefore $C = \prod_x C_x$ is compact.

$\prod_{x \in X} C_x =$ set of all functions mapping $X$ to the complex plane with the property that $|F(x)| \leq \|x\|$. Hence the unit sphere of $X^*$ can be regarded as a subspace and so will be compact as it is closed.

**Theorem** If $X$ is a Banach space, $X$ is reflexive $\Leftrightarrow$ its unit sphere is weakly compact.

**Proof** If $X$ is reflexive $X^{**} = X$ the weak topology of the unit sphere of $X$ is the same as the weak $*$ topology which is compact by th previous theorem.

**Closed Graph Theorem** $T$ linear




Let $G(T) = \{(X, T(X)\} \subset X \times Y$. If $T$ is continuous $G(T)$ is closed. The theorem states that the converse is true.

**Lemma** Let $T$ be a bounded linear transformation of a Banach space $X$ into a Banach sphere $Y$. If the image under $T$ of the unit sphere $S_1 = S(0,1)$ is dense in some sphere $U_r = S(0,r)$ about the origin of $Y$ than it includes the whole of $U_r$.

**Proof** Let $\delta > 0$. We wish to define a sequence $\{y_n\}$ such that

$$y_{n+1} - y_n \in \delta^n S, \ \|y_{n+1} - \overline{y}\| < \delta^{n+1} r \tag{1}$$

We define $y_0 = 0$. Suppose that $y_0 \ldots y_n$ have been defined.

Since $\overline{y} \in S(\overline{y}, \delta^{n+1} r) \cap (y_n + \delta^n U_r)$ this is a non-empty open subset of $y_n + \delta^n U_r$ therefore there is an element $y_{n+1}$ of $y_n + \delta^n T(S_1)$ which belongs to this set, and $y_{n+1}$ satisfies the conditions (1).

$y_n \to \overline{y}$ as $n \to \infty$ provided $\delta < 1$.

$\exists x_n$ such that $x_n \in \delta^n S_1$ and $T(x_n) = y_{n+1} - y_n$.

We can define $\overline{x} = \sum_1^\infty x_n$ provided $\delta < 1$. Since $T$ is bounded

$$\begin{aligned} T(\overline{X}) &= \lim_{N\to\infty} \sum_1^N T(x_n) \\ &= \lim_{N\to\infty} y_{N+1} = \overline{y} \\ \|\overline{x}\| &\leq |sum\delta^n = \frac{1}{1-\delta} \text{ for all} \delta \\ \text{therefore } \|\overline{x}\| &\leq 1 \\ U_{r(1-\delta)} &\subset T(S_1) \text{ for every } \delta \\ \text{and } U_r &= \cup_\delta U_{r(1-\delta} \subset T(S_1) \end{aligned}$$

**Proof of Theorem** Let $N(x) = \|x\| + \|T(x)\|$

If $\{x_n\}$ is a Cauchy sequence for the norm $N$ then it is a Cauchy sequence for $\|X\|$ and also $\{T(x_n)\}$ is a Cauchy sequence for $\|T_x\|$.

Therefore $x_n \to x$ and $Tx_n \to y$ as $n \to \infty$ as $G(T)$ is closed $(x, y) \in G(T)$ therefore $y = T(x)$.

$$N(x - x_n) = \|x - x_n\| + \|Tx - Tx_n\| \to 0 \text{ as } n \to \infty$$

Therefore $X$ is a Banach space for the new norm $N$. Now the identity mapping from $X$ with norm $N$ to $(X, \|\ \|)$ is bounded since $\|X\| \leq N(x)$.

If $S_1$ denotes the unit sphere defined by $N$, it follows from Baire's category theorem that $S_1$ is dense in some sphere $U_r$ about the origin defined by $\|\ \|$.

Thus by the lemma applied to the identity mapping $U_r \subset S_1$

i.e. if $\|X\| < r \Rightarrow N(x) < 1$

i.e. $N(X) \leq \frac{1}{r}\|x\|$

so $\|T(x)\| \leq N(x) \leq \frac{1}{r}\|x\|$ and so $T$ is continuous.

**Hilbert Space** A pre-Hilbert Space is a real or complex vector spade in which an inner product $(x, y)$ is defined having the following properties.

**(i)** $(x, x) > 0$ unless $x = 0$

**(ii)** $(x,y) = \overline{(y,x)}$

**(iii)** $(x+y,z) = (x,z) + (y,z)$

**(iv)** $\lambda x, y) = \lambda(x,y)$

A pre-Hilbert space can be normed by defining $\|x\| = (x,x)^{\frac{1}{2}}$.

A Hilbert space is a pre-Hilbert space which is complete for this norm.

A Banach space is a Hilbert space

$$\Leftrightarrow \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

**Schwarz inequality** $|(x,y)| \leq \|x\|\ \|Y\|$

**Proof**

$$\begin{aligned}(\lambda x - y, \lambda x - y) &= |\lambda|^2\|x\|^2 - 2R\lambda(x,y) + \|y\|^2 \\ 2R\lambda(x,y) &\leq |\lambda|^2\|x\|^2 + \|y\|^2\end{aligned}$$

Choose $\lambda$ so that $|\lambda| = \frac{\|Y\|}{\|X\|}$ and $\arg\lambda = -\arg(x,y)$.

$$2\frac{\|Y\|}{\|X\|}|(x,y)| \leq 2\|y\|^2$$

Hence the result.

**Minkowski Inequality** $\|x+y\| \leq \|x\| + \|y\|$

**Proof**

$$\begin{aligned}|\|x+y\|^2| &= |(x+y, x+y)| \\ &= |\|X\|^2 + 2R(x,y) + \|y\|^2| \\ &\leq \|x\|^2 + \|Y\|^2 + 2|(x,y)| \leq (\|x\| + \|Y\|)^2\end{aligned}$$

using Schwarz.

**Theorem** A closed convex subset $C$ of a Hilbert space contains a unique element of smallest norm.

**Proof** Let $d = \inf\{\|x\| : x \in C\}$.

Then $\exists\{x_n\} \subset C$ such that $\|x_n\| \to d$. Since $C$ is convex $\frac{x_n + x_m}{2} \in C$ therefore $\|x_n + x_m\| \geq 2d$.

$$\begin{aligned}\|x_n - x_m\|^2 &= 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2 \\ &\leq 2\{\|x_n\|^2 - d^2\} + 2\{\|x_m\| - d^2\} < \varepsilon\end{aligned}$$

if $n$ and $m$ are sufficiently large.

Hence the sequence is a Cauchy sequence and has a limit point $x$ which belongs to $C$ as $C$ is closed, and $\|x\| = d$.

If $y \in C$ and $\|y\| = d$ then $\|x + y\| \geq 2d = \|x\| + \|y\|$ and so $y = \lambda x$ where $\lambda > 0 \Rightarrow \|y\| = \lambda\|x\| \Rightarrow \lambda = 1$ therefore $x = y$.

**Theorem** Let $M$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then any $x = x_1 + x_2$ where $x_1 \in M$ and $x_2$ perpendicular $M$ (i.e. $(x_2, y) = 0$ for all $y \in M$).

**Proof** Suppose $x \in M$. Let $x_2$ be the element in the closed convex set $x + M$ which is closest to 0.

Put $x_1 = x - x_2 \in M$.

If $y \in M$ then for any scalor $\lambda$

$$\|x_2 + \lambda y\|^2 \geq \|x_2\|^2$$

Since $2R\overline{\lambda}(x_2, y) + |\lambda|^2\|y\|^2 \geq 0$

Put $\lambda = -\frac{(x_2\ y)}{\|y\|^2}$.

Then $-\frac{|(x_2\ y)|^2}{\|y\|^2} \geq 0$ therefore $(x_2\ y) = 0$.

Suppose $x = x_1' + x_2' = x_1 + x_2$ therefore $x_1 - x_1' = x_2' - x_2 = 0$.

Hence uniqueness.

If $M$ is closed $\mathcal{H} = M + M^\perp$

If $M$ is closed and $x \in M^{\perp\perp}$

$$\begin{aligned} x &= x_1 + x_2 \quad x_1 \in M \ x_2 \in M^\perp \\ (x\ x_2) &= (x_1\ x_2) + (x_2\ x_2)\end{aligned}$$

Therefore $(x_2\ x_2) = 0$ therefore $x_2 = 0$ therefore $x \in M$.

**Theorem** Suppose $\mathcal{H}$ is any Hilbert Space and let $f \in X^*$. Then there is an element $y \in H$ such that $f(x) = (x, y)$ for every $x \in \mathcal{H}$.

**Proof** Let $M$= null space of $f$.

$\exists y_0 \perp M$ such that if $x \in \mathcal{H}$

$$\begin{aligned} x &= m + \lambda y_0 \; m \in M \\ f(x) &= \lambda f(y_0) \\ (x,\; y_0) &= \lambda \|y_0\|^2 \\ f(x) &= \frac{(x,\; y_0)}{\|y_0\|^2} f(y_0) = \left(x, \frac{\overline{f(y_0)}}{\|y_0\|^2} y_0\right) \end{aligned}$$

Write $y = \frac{\overline{f(y_0)}}{\|y_0\|^2} y_0$.

If $M$ is any closed subspace and $x \in \mathcal{H}$

$$\begin{aligned} x &= x_1 + x_2 \; x_1 \in M \; x_2 \in M^\perp \\ x_1 &= \text{Proj}_M x \end{aligned}$$

If $T(x) = x_1$ $T$ is a linear operator from $\mathcal{H}$ to itself, and $\|T\| = 1$.

$$\begin{aligned} TT' &= T \\ (Tx, y) = (x_1\; y) = (x_1\; y_1) \\ (x, Ty) &= (x\; y_1) = (x_1\; y_1) \end{aligned}$$

Therefore $T$ is self-adjoint.

**Theorem** If $M_1, \ldots, M_n$ are $n$ mutually perpendicular closed subspaces of a Hilbert space $\mathcal{H}$ and if $x \in \mathcal{H}$ and $x_i, \ldots, x_n$ are the projections of $x$ on $M_1, \ldots M_n$ respectively, then

$$\sum \|x_i\|^2 leq \|x\|^2$$

**Proof** Put $M = M_1 + +M_n$ $x = x_1 + \ldots + x_n + y,\; y \in M^\perp$, then $\|x\|^2 = \sum \|x_1\|^2 + \|y\|^2$.

**Theorem** Let $\{M_\alpha\}$ be a family, possibly uncountable, of pairwise orthogonal closed subspaces of $\mathcal{H}$, and let $M$ be the closure of their direct sum.

If $x_\alpha = \text{proj}_{M_\alpha} x \; x \in \mathcal{H}$ then $x_\alpha = 0$ except for a countable set of indices $\alpha_n$.

$\sum x_{\alpha_n}$ is convergent and its sum is the projection of $x$ on $M$.

**Proof**

$$\sum_{i=1}^{r} \|x_{\beta_i}\| \leq \|x\|^2$$

Hence for any $n$ the number of indices satisfying $\|x_\alpha\| \geq \frac{1}{n}$ is finite therefore the number of indices satisfying $\|x_\alpha\| > 0$ is countable.

$\sum_1^N \|x_{\alpha_n}\|^2 \leq \|x\|^2$ for each $N$ therefore $\sum_1^\infty \|x_{\alpha_n}\|^2 < +\infty$.

If $y_n = \sum_1^N x_{\alpha_n}$

$$\|y_n - y_m\|^2 \leq \sum_{m+1}^{n} \|x_{\alpha_i}\|^2 < \varepsilon$$

if $m$ is sufficiently large. Therefore $\{y_n\}$ is a Cauchy sequence which tends to a limit $y = \sum_1^\infty x_{\alpha_n}$ in $M$, as $M$ is closed.

It remains to prove that $x - y \perp M$.

It is sufficient to prove that

$$w_{\beta_1} + w_{\beta_2} + \ldots + w_{\beta_r} \perp x - y$$

where $w_{\beta_i} \in M_{\beta_1}$ as the class of all such vectors is everywhere dense in $M$.

If $\beta_1$ as an $\alpha_n$

$$\begin{aligned}(x - y, w_{\beta_1}) &= (x \; w_{\beta_1}) - (x_{\beta_1} \; w_{\beta_1}) \\ &= (x_{\beta_1} \; w_{\beta_1}) - (x_{\beta_1} \; w_{\beta_1}) = 0\end{aligned}$$

If $\beta_1$ is not an $\alpha_n$ then $w_{\beta_1} \perp x$ and $\perp y$ and so to $x - y$.

**Orthonormal vectors** A set $N$ of vectors in a Hilbert space $\mathcal{H}$ is said to be orthonormal if $\|x\| = 1$ for every $x$ in $N$, and $(x, y) = 0$ for all $y$ in $N \neq x$.

An orthonormal set $N$ of vectors is conplete if $N^{\perp} = \{0\}$.

Let $M_x$ be the 1-dimensional subspace generated by $x$ in $N$.

If $y \in \mathcal{H}$

$$\mathrm{proj}_{M_x} y = \frac{(y\ x)}{\|x\|}.x = (y, x).x$$

as $\|x\| = 1$ $(y, x) = 0$ except for a sequence $\{x_n\}|subsetN$ and for this sequence

$$\begin{aligned} y &= \sum (y\ x_n) x_n \\ \|y\|^2 &= \sum |(y\ x_n)|^2 \end{aligned}$$

This condition of completeness is equivalent to:

**(i)** for any $y$ in $\mathcal{H}$ $y = \sum_{x \in N} (y\ x) x$

**(ii)** for any $y$ in $\mathcal{H}$ $\|y\|^2 = z \sum_{x \in N} |(y\ x)|^2$.