**Question**

In the Poincáre disc $\mathbf{D}$, consider the parallelogram $P_s$ bounded by the four hyperbolic lines

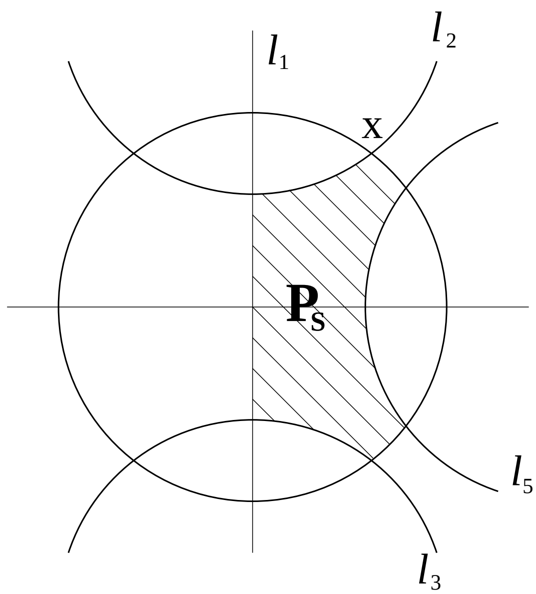
$$\ell_1 = \{z \in \mathbf{D} : \mathrm{Re}(z) = 0\}$$
$$\ell_2 = \{z \in \mathbf{D} : |z - 2i| = \sqrt{3}\}$$
$$\ell_3 = \{z \in \mathbf{D} : |z + 2i| = \sqrt{3}\}$$

and

$$\ell_s = \{z \in \mathbf{D} : |z - s| = \sqrt{s^2 - 1}\}$$

where$s$ is real and $s > 1$. Determine the values of $s$ for which $P_s$ has finite hyperbolic area.

**Answer**




$P_s$ has finite area when $\ell_s$ and $\ell_2$ (or $\ell_s$ and $\ell_3$ by symmetry) are parallel but not ultraparallel (that is, when $\ell_2\ell_s$ intersect at the boundary $S^1$ at infinity of $\mathbf{D}$).

Since this occurs when $\ell_2\ell_s$ are tangent, the distance between their centers is equal to the sum of their radii, and so

$|s - 2i| = \sqrt{3} + \sqrt{s^2 - 1}$

$\sqrt{s^2 + 4} = \sqrt{3} + \sqrt{s^2 - 1}$

Squaring:

$s^2 + 4 = 3 + s^2 - 1 + 2\sqrt{3}\sqrt{s^2 - 1}$

$2 = 2\sqrt{3}\sqrt{s^2 - 1}$

$s^2 - 1 = \frac{1}{3}, \quad s^2 = \frac{4}{3}, \quad s = \frac{2}{\sqrt{3}}$ and so $P_s$ has finite hyperbolic area for $s \geq \frac{2}{\sqrt{3}}$.