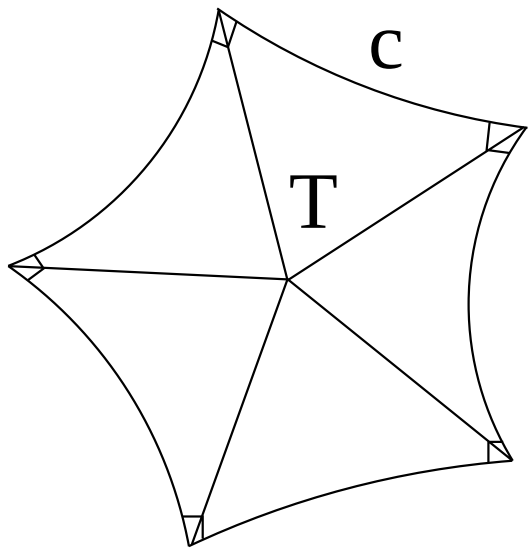
Question

Calculate the perimeter of the regular hyperbolic pentagon **R**, all of whose interior angles are $\frac{\pi}{2}$.

Answer



If we are to draw R in the Poincáre Disc, we could do so so that the vertices are at $re^{i\frac{2\pi k}{5}}$ $(0 \leq k \leq 4)$ for some $0 < r < 1$. Joining the angles to the vertices then breaks R into 5 triangles with interior angles $\dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{2\pi}{5}$.
Let T be one of these triangles; and let c be the side opposite the vertex with angle $\dfrac{2\pi}{5}$. Then, by the law of cosines II:

$$\begin{aligned}\cosh(c) &= \frac{\cos(\frac{\pi}{4})\cos(\frac{\pi}{4}) + \cos(\frac{2\pi}{5})}{\sin(\frac{\pi}{4})\sin(\frac{\pi}{4})} \\ &= \frac{\frac{1}{2} + \cos(\frac{2\pi}{5})}{\frac{1}{2}} = x\end{aligned}$$

Then, $\dfrac{1}{2}(e^c + e^{-c}) = x$, so $\begin{array}{l} e^c + e^{-c} - 2x = 0 \\ e^{2c} - 2xe^c + 1 = 0 \end{array}$

$$\begin{aligned}e^c &= \frac{1}{2}(2x \pm \sqrt{4x^2 - 4}) \\ &= x + \sqrt{x^2 - 1}\end{aligned}$$

So perimeter of R is $\underline{5c = 5\ln(x + \sqrt{x^2 - 1})}$
where $\underline{x = 1 + 2\cos(\frac{2\pi}{5})}$.