Question
Determine whether or not the four points $1,\ -3,\ -1,\ -2i$, and $3i$ lie on a circle in the Riemann sphere $\overline{C}$.

Answer
One approach is to use the cross ratio. Another is to proceed directly, using that Möbius transformations take circles to circles. Specifically, construct the Möbius transformation taking 1 to , -3 to ∞, $3i$ to 0, and see what this transformation does to $-1-2i$; these 4 points lie on a circle if and only if $-1-2i$ goes to a point on $\mathbf{R}$.

$m(-3)=\infty$ and $m(3i)=0$ give $m(z)=\dfrac{z-3i}{z+3}$

$m(1)=1$ gives $m(z)=\dfrac{z-3i}{z+3}\cdot\dfrac{4}{1-3i}$

Then,

$$\begin{aligned} m(-1-2i) &= \frac{-1-2i-3i}{-1-2i+3}\cdot\frac{4}{1-3i} \\ &= \frac{-4-20i}{(2-2i)(1-3i)} \\ &= \frac{-4-20i}{-4-8i}\cdot\frac{-4+8i}{-4+8i} \\ &= \frac{16+160+80i-32i}{80} \end{aligned}$$

which is not real.
So, these points <u>do not lie on a circle in $\overline{\mathbf{C}}$</u>.