Question
Let X have pdf $f(x) = \frac{1}{2}(1+x), \quad -1 < x < 1$. Find the pdf of $Y = X^2$.

Answer
$f(x) = \frac{1}{2}(1+x), \quad -1 < x < 1$.
The transformation is $y = x^2$. It is decreasing in $-1 < x \leq 0$ and increasing in $0 < x < 1$.
Also $0 < y < 1$ and $x = \pm\sqrt{y}$
Therefore $\left|\dfrac{dx}{dy}\right| = \dfrac{1}{2\sqrt{y}}$
The pdf of Y is

$$\begin{aligned} g(y) &= \frac{1}{2}(1-\sqrt{y})\cdot\frac{1}{2\sqrt{y}} + \frac{1}{2}(1+\sqrt{y})\cdot\frac{1}{2\sqrt{y}}, \quad 0 < y < 1 \\ &= \frac{1}{4\sqrt{y}}.2, \quad 0 < y < 1 \\ &= \frac{1}{2\sqrt{y}}, \quad 0 < y < 1. \end{aligned}$$

Alternative:

$$\begin{aligned} F(x) &= \int_{-1}^{x} f(t)\,dt \\ &= \frac{1}{2}\int_{-1}^{x}(1+t)\,dt \\ &= \frac{1}{2}\left[t + \frac{t^2}{2}\right]_{-1}^{x} \\ &= \frac{1}{2}\left\{x + \frac{x^2}{2} + 1 - \frac{1}{2}\right\} \\ &= \frac{1}{2}\left\{x + \frac{x^2}{2} + \frac{1}{2}\right\} \end{aligned}$$

$$\begin{aligned} G(y) &= P(Y \leq y) \\ &= P\{x^2 \leq y\} \\ &= P\{-\sqrt{y} \leq x \leq \sqrt{y}\} \\ &= F(\sqrt{y}) - F(-\sqrt{y}) \\ &= \frac{1}{2}\left\{\sqrt{y} + \frac{y}{2} + \frac{1}{2} + \sqrt{y} - \frac{y}{2} - \frac{1}{2}\right\} \\ &= \sqrt{y} \end{aligned}$$

Therefore the pdf of Y is $g(y) = \dfrac{dG(y)}{dy} = \dfrac{1}{2\sqrt{y}}, \quad 0 < y < 1.$