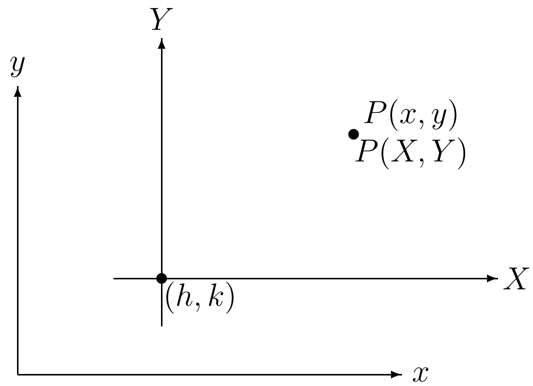
## Coordinate Geometry

### Equations of Second degree

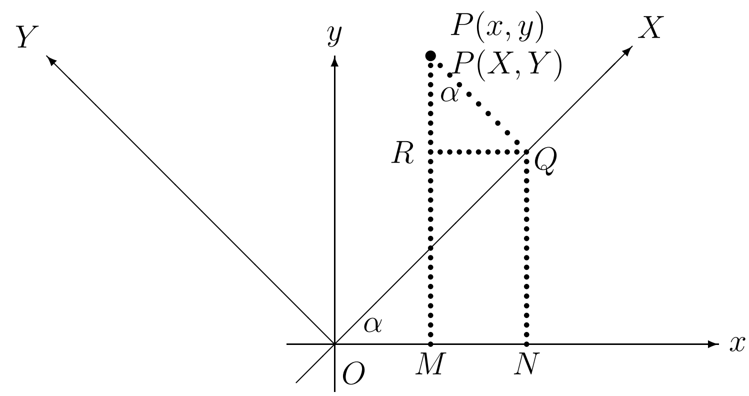
The problem dealt with here is the following, given an expression quadratic in $x$ and $y$, what curve does it represent in rectangular cartesian co-ordinates. To answer this question we shall need to be able to translate and rotate axes in cartesian co-ordinates.

Translation




$$x = X + k$$
$$y = Y + k$$
$$\updownarrow$$
$$X = x - h$$
$$Y = y - k$$

Rotation




$x = OM = ON - MN = ON - QR = X\cos\alpha - Y\sin\alpha$

$y = MP = MR + RP = QN + RP = X\sin\alpha + Y\cos\alpha$

$$\begin{array}{c} x = X\cos\alpha - Y\sin\alpha \\ y = X\sin\alpha + Y\cos\alpha \end{array} \longleftrightarrow \begin{array}{c} X = x\cos\alpha + y\sin\alpha \\ Y = y\cos\alpha - x\sin\alpha \end{array}$$

Using these transformations it can be shown that the following cases occur. The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ represents

I) A parabola (including the degenerate case of parallel lines) if $B^2 - 4AC = 0$.

II) An ellipse, or possibly no real locus, if $B^2 - 4AC < 0$. The ellipse is a circle if $B = 0$ and $A = C$. It may degenerate to a point.

III) A hyperbola (or a pair of intersecting straight lines) if $B^2 - 4AC > 0$.

To reduce an equation to a standard form

I) rotate the axes through angle $\alpha$, to make the product term vanish $\left(\tan 2\alpha = \dfrac{B}{A-C}\right)$

II) and III) First translate the origin so that the terms of degree 1 vanish. Then rotate the new axes through angle $\alpha$, to make the product term vanish $\left(\tan 2\alpha = \dfrac{B}{A-C}\right)$

For proofs of the above assertions see D.S.Jones and D.W.Jordan Introductory Analysis Volume 1 pp54-58.

Example

i) Investigate the conic $3x^2 + 5xy - 2y^2 - x + 5y - 2 = 0$ (1)

$B^2 - 4AC = 25 + 24 = 49 > 0$

So we have a hyperbola or a pair of lines.

Put $x = X + h \quad y = Y + k$

Equation (1) then becomes

$3(X+h)^2 + 5(X+h)(Y+k) - 2(Y+k)^2 - (X+h) + 5(Y+k) - 2 = 0$

$3X^2 + 5XY - 2Y^2 + X(6h + 5k - 1) + Y(5h - 4k + 5) + 3h^2$
$\quad +5hk - 2k^2 - h + 5k - 2 = 0 \quad (2)$

We choose $h, k$ to satisfy

$6h + 5k - 1 = 0 \qquad 5h - 4k + 5 = 0$

so $h = -\frac{3}{7} \qquad k = \frac{5}{7}$

Equation (2) then becomes $3X^2 + 5XY - 2Y^2 = 0$

$(3X - Y)(X + 2Y) = 0$

so it is a pair of straight lines.

ii) $xy + 2x + y = 0$ $\qquad B^2 - 4AC = 1 > 0$

so we have a hyperbola or a pair of lines.

$x = X + h \qquad y = Y + k$ gives

$(X + h)(Y + k) + 2(X + h) + (Y + k) = 0$

$XY + X(k + 2) + Y(h + 1) + hk + 2h + k = 0$

we choose $h = -1 \qquad k = -2$ so $XY - 2 = 0$. A rectangular hyperbola.

Now rotate through $\alpha$

$X = \xi \cos\alpha - \eta \sin\alpha \qquad Y = \xi \sin\alpha + \eta \cos\alpha$

$XY = (\xi \cos\alpha - \eta \sin\alpha)(\xi \sin\alpha + \eta \cos\alpha)$

$$= \xi^2 \frac{1}{2} \sin 2\alpha - \eta^2 \frac{1}{2} \sin 2\alpha + \xi\eta \cos 2\alpha = 2$$

choose $\alpha$ so that $\cos 2\alpha = 0$

so $2\alpha = \frac{\pi}{2} \qquad \alpha = \frac{\pi}{4}$

$XY - 2 = 0$ becomes $\xi^2 - \eta^2 = 4$

DIAGRAM