Coordinate Geometry

Conic sections

These are plane curves which can be described as the intersection of a cone with planes oriented in various directions.
It can be demonstrated that the locus of a point which moves so that its distance from a fixed point (the focus) is a constant multiple (e - the eccentricity) of its distance from a fixed straight line (the directrix) is a conic section.
If $e < 1$ we obtain an ellipse.
If $e = 1$ we obtain a parabola.
If $e > 1$ we obtain a hyperbola.
See Scientific American September 1977-Mathematical games section p24.

Cartesian equation

Take as the x-axis a line perpendicular to the directrix passing through the focus. Take the origin to be where the conic cuts the axis between the focus and directrix.
DIAGRAM
From the definition of a conic $SP^2 = e^2PM^2$
$y^2 + (x - ek)^2 = e^2(x + k)^2$
$y^2 + x^2 - 2ekx + e^2k^2 = e^2x^2 + 2e^2kx + e^2k^2$
$y^2 + x^2\left(1 - e^2\right) - 2ke(1 + e)x = 0$
If we have a parabola where $e = 1$ then the equation reduces to $y^2 = 4kx$.
If $e \neq 1$ we write the equation in the form

$$\frac{y^2}{1 - e^2} + \left(x - \frac{ke}{1 - e}\right)^2 = \frac{k^2e^2}{(1 - e)^2}$$

We now write $\dfrac{ke}{1 - e} = a$, and shift the origin to the point $(a, 0)$. Referred to these new axes the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2\left(1 - e^2\right)} = 1$$

The focus becomes the point $(-ae, 0)$ and the directrix the line $x = -\dfrac{a}{e}$.
Notice that the equation is unchanged if x is replaced be $-x$, so that there is a second focus at $x = (ae, 0)$ and a second directrix at $x = \frac{a}{e}$.
For an ellipse $e < 1$ and we write $b^2 = a^2(1 - e^2)$ so the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

For a hyperbola $e > 1$ and we write $b^2 = a^2(e^2 - 1)$ so the equation becomes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Focal distance properties

Ellipse ($e < 1$)

DIAGRAM

From the definition

$$\begin{aligned} S_1P + S_2P &= ePM_1 + ePM_2 = e(PM_1 + PM_2) \\ &= eM_1M_2 = e\frac{2a}{e} = 2a \end{aligned}$$

So the sum of the focal distances is constant.

Hyperbola ($e > 1$)

DIAGRAM

From the definition

$$\begin{aligned} S_2P - S_1P &= ePM_2 - ePM_1 = e(PM_2 - PM_1) \\ &= e\frac{2a}{e} = 2a \end{aligned}$$

Similarly

$S_1Q - S_2Q = 2a$

The Parabolic Mirror

DIAGRAM

Suppose a ray of light comes in parallel to the x-axis and is reflected in a direction equally inclined to the tangent. We prove that it passes through the focus.

Let the parabola have equation

$y^2 = 4kx$, so $S = (k, 0)$, $\quad P = (x, y)$

$2y\frac{dy}{dx} = 4k$ so $\frac{dy}{dx} = \frac{2k}{y}$

thus $\tan\alpha_1 = \frac{2k}{y}$.

Now $\tan\alpha_3 = \frac{y}{x-k}$ and $\alpha_2 = \alpha_3 - \alpha_1$

So $\tan\alpha_2 = \tan(\alpha_3 - \alpha_1) = \frac{\tan\alpha_3 - \tan\alpha_1}{1 + \tan\alpha_3\tan\alpha_1} = \frac{2k}{y}$ (verify)

so $\alpha_1 = \alpha_2$
So a parallel beam of light will be reflected through the focus.

Parametric equations

Because a curve is one-dimensional we can label the points by means of a single real variable, as in the following examples. Traditionally the letter t is used as the parameter, analogous with the curve being traced out in time.

Examples

i) $x = a + t, \quad y = b + mt$ represents the straight line through (a, b) with slope m.

ii) $x = a\cos t, \quad y = a\sin t$ represents the circle of radius a centred at (0,0). We use $\cos^2 + \sin^2 = 1, \quad t$ corresponds to an angle and so θ is sometimes used.

iii) $x = a\cos t, \quad y = b\sin t$ represents the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ again t represents an angle but not the angle from O to P.

DIAGRAM

iv) to parameterise a hyperbola we need to find $\dfrac{x}{a} = f(t), \quad \dfrac{y}{b} = g(t)$ so that $f(t)^2 - g(t)^2 = 1$.

There are several possibilities

a) $\dfrac{x}{a} = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right) \quad \dfrac{y}{b} = \dfrac{1}{2}\left(t - \dfrac{1}{t}\right)$

b) $x = a\sec t \qquad y = b\tan t$

c) $\left.\begin{array}{l} \dfrac{x}{a} = \dfrac{1}{2}\left(e^t + e^{-t}\right) = \cosh t \\ \dfrac{y}{b} = \dfrac{1}{2}\left(e^t - e^{-t}\right) = \sinh t \end{array}\right\}$ These are called hyperbolic functions.

v) to parameterise the parabola $y^2 = 4kx$ we use $x = kt^2, \quad y = 2kt$

DIAGRAM

As t increases this induces a direction on the curve.

The curve described in the opposite direction can be parameterised by $x = kt^2 \quad y = -2kt$.

DIAGRAM

We regard these two as different curves (with the same set of points).

It is important to distinguish the direction in many applications.

Polar equation of a conic

We want to find the polar equation of a conic with the origin as focus.
DIAGRAM
DIAGRAM
$PS = ePM \quad \sqrt{x^2 + y^2} = e(x + k(e + 1)) \qquad (1)$
Converting to polars gives
$r = er\cos\theta + ek(e + 1)$
notice that from (1) $ek(e + 1)$ is the y-value when $x = 0$
DIAGRAM
Write $l = ek(e + 1)$. The length $2l = PP'$. PP' is called the latus rectum.
l is the semi-latus rectum.
Thus we can write the conic as

$$\frac{l}{r} = 1 - e\cos\theta$$

Note that rotations are easy in polar co-ordinates, so the equation

$$\frac{l}{r} = 1 - e\cos(\theta - \alpha)$$

is a conic having its axis at an angle α with the initial line. Notice that when $\alpha = \pi$ the equation becomes

$$\frac{l}{r} = 1 + e\cos\theta$$

In the case of an ellipse or hyperbola this is equivalent to using the other focus as an origin.
Notice that if $e > 1$ we can sometimes have $\frac{l}{r} < 0$. Although we normally insist on $r > 0$ in polars, in interpreting polar equations it is often convenient to allow $r < 0$, meaning r measured in the other direction through O.
e.g.

$$\frac{1}{r} = 1 - 2\cos\theta$$

when $\theta = 0$ this gives $\frac{1}{r} = -1, r = -1$.
We plot $\theta = 0, r = -1$ as the point $(-1, 0)$.

When $\cos\theta = \frac{3}{4}$, $(\theta \approx 41^\circ)$ this gives $\frac{1}{r} = -\frac{1}{2}, r = -2$