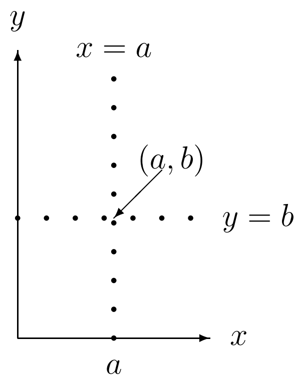
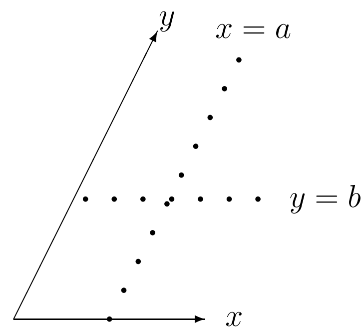
Coordinate Geometry

In this section we shall discuss various co-ordinate systems in 2 and 3 dimensions and equations for a variety of curves.

Cartesian coordinates are formed by two sets of parallel lines, usually orthogonal, although not necessarily so. A point is specified by an ordered pair (a, b), and can be thought of as the intersection of the lines $x = a$ and $y = b$.



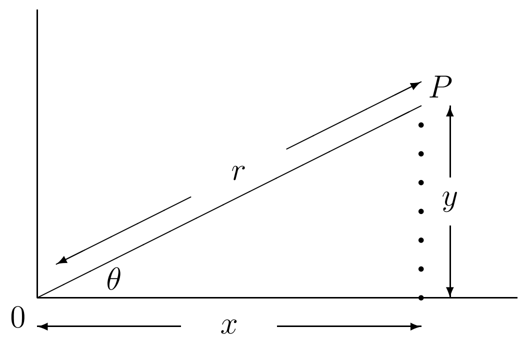


You will be familiar with many equations and the corresponding curves in rectangular cartesian co-ordinates. You should note that the shape of a curve connected with an equation depends on the co-ordinate system.
e.g. $x^2 + y^2 = 1$ is the equation of a circle in rectangular cartesian co-ordinates, but in oblique cartesian co-ordinates it represents an ellipse.

Polar Co-ordinates

This system is set up with an origin O and a direction fixed through O. The co-ordinates of a point P are specified as the distance OP and the angle OP makes with the fixed direction. Thus we have $P(r, \theta)$. Each pair of numbers specifies a point, but without any restrictions on r and θ a point may be assigned more than one set of co-ordinates.
e.g $(1, 0)$ $(1, 2\pi)$
If we interpret negative r as reflected through the origin then $(1, 0)$ $(-1, \pi)$.
In order to achieve uniqueness we make the restriction $r > 0$, and we restrict θ to belong to some interval of length 2π. e.g $-\pi < \theta \leq \pi$ is the most common. In describing some curves with more than one point in a given

direction like spirals it is however convenient to relax this restriction on θ. We shall always take $r \geq 0$ however. There is still a slight problem about the origin O. Clearly $r = 0$, but what about θ? We do not specify co-ordinates uniquely for O, but if we obtain $r = 0$ from an equation then that will correspond to O.
Just as the grid for Cartesian co-ordinates consists of lines x=const, y=const so the grid for polar co-ordinates consists of r=const (circles centre O), θ=const (half-lines starting at O).
We can transform from cartesian to polar co-ordinates as follows.



If P has cartesian co-ordinates (x, y) and polar co-ordinates (r, θ) then
$r = \sqrt{x^2 + y^2}$, $\tan\theta = \dfrac{y}{x}$, $x = r\cos\theta$, $y = r\sin\theta$.
Note that it is ambiguous to write $\theta = \tan^{-1}\dfrac{y}{x}$

since if $(x, y) = (1, 1)$ $\quad \frac{y}{x} = 1$ and $\theta = \frac{\pi}{4}$
and if $(x, y) = (-1, -1)$ $\quad \frac{y}{x} = 1$ but $\theta = -\frac{3\pi}{4}$

We could write unambiguously $\theta = \tan^{-1}\frac{y}{x}$ where $0 < \theta < \pi$ if $y > 0$ and $-\pi < \theta < \pi$ if $y < 0$

(We still need the special cases $x = 0$, $y = 0$ for completeness) but it is better to draw a diagram as well.

Notice that to say "the point P has co-ordinates (a, b)" is ambiguous out of context. It depends what co-ordinate system we are using.

So rectangular cartesia	$P = (2, 1)$
60° oblique, same O and $x$axis	$P = (2 - \frac{1}{\sqrt{3}})$
polars, same O and $x$axis	$P = (\sqrt{5}, 0.464)$ (radians)

Some Equations in Polar Co-ordinates

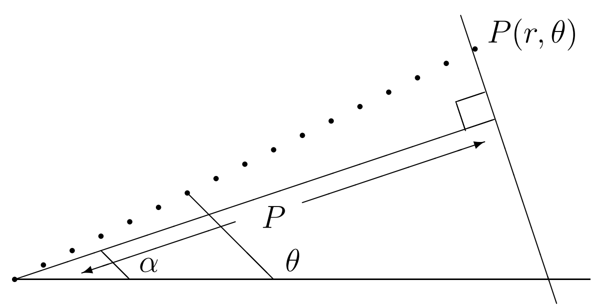
i) Straight line

Since the cartesian equation is $ax + by + c = 0$

we use $x = r\cos\theta$, $y = r\sin\theta$ to obtain

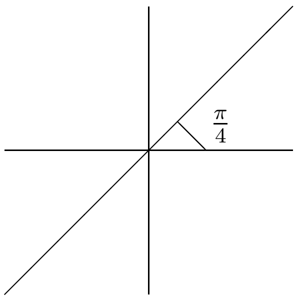
$$r(a\cos\theta + b\sin\theta) + c = 0 \quad (*)$$

alternatively we have $r\cos(\theta - \alpha) = p$ (*)

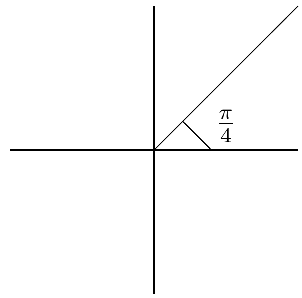


You should try and relate the two forms of equation above.

Note that the equation $\theta = \frac{\pi}{4}$ does not represent the whole line



but only



You should analyse what happens to the equations (*) when the line passes through O.

ii) circle

A circle centred at O has equation $r = a$.

A circle having O on the circumference and the initial line as diameter.

DIAGRAM

$r = a \cos \theta$

DIAGRAM

$r = a \cos(\theta - \alpha)$

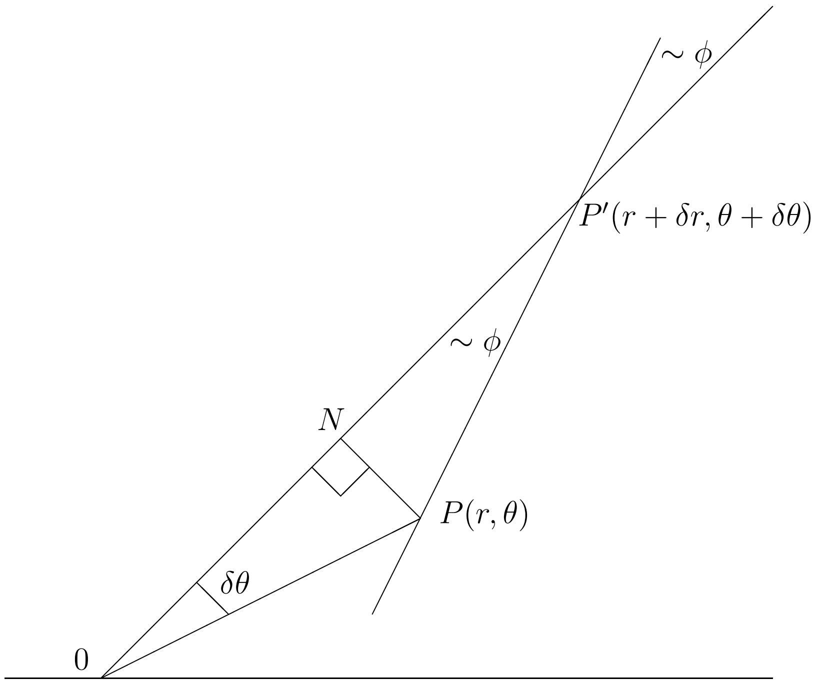
DIAGRAM

Applying the cosine formula to the triangle POC gives

$$a^2 = r^2 + d^2 - 2rd \cos(\theta - \alpha).$$

Again you should convert from cartesians to polars and try to relate the equations.

We now consider slopes of tangents in polar co-ordinates.

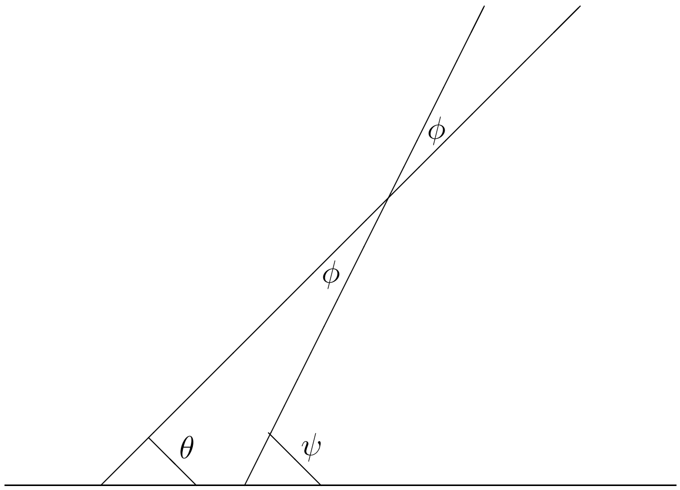


$\tan\phi \approx \frac{PN}{NP'} \approx \frac{r\delta\theta}{\delta r} \approx r\frac{d\theta}{dr}$
$PN = OP \sin PON = r \sin\delta\theta$

$$\begin{aligned} P'N &= OP' - ON = r + \delta r - r\cos\delta\theta = \delta r + r(1-\cos\delta\theta) \\ &= \delta r + r * 2\sin^2\frac{1}{2}\delta\theta \end{aligned}$$

$$\begin{aligned} \tan OP'P &= \frac{PN}{P'N} = \frac{rsin\delta\theta}{\delta r + r(2\sin^2\frac{1}{2}\delta\theta)} = \frac{r*2\sin\frac{1}{2}\delta\theta\cos\frac{1}{2}\delta\theta}{\delta r + r(2sin^2\frac{1}{2}\delta\theta)} \\ &= \frac{r\frac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}\frac{\delta\theta}{\delta r}\cos\frac{1}{2}\delta\theta}{1 + r\frac{sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}\frac{\delta\theta}{\delta r}\sin\frac{1}{2}\delta\theta} \\ &\rightarrow \frac{r*1*\frac{d\theta}{dr}*1}{1+r*1*\frac{d\theta}{dr}*0} = r\frac{d\theta}{dr} \quad \text{as } P' \rightarrow P \end{aligned}$$

$OP'P$ tends to the angle ϕ shown below



$\tan\phi = r\frac{d\theta}{dr}$, $\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$

You should check through the above in the case when

$\delta r < 0$, or $OP' < ON$.

Example

Consider the circle $r = a\cos\theta$

DIAGRAM

$$\frac{dr}{d\theta} = -a\sin\theta \qquad \frac{1}{r}\frac{dr}{d\theta} = -\tan\theta = \cot\phi$$

So $-\tan\theta = \tan(\frac{\pi}{2} - \phi) = -\tan(\phi - \frac{\pi}{2})$

so $\theta = \phi - \frac{\pi}{2}$ or $\phi = \frac{\pi}{2} + \theta$.

Consider the equation $r = a(1 + \cos\theta)$. Because $\cos\theta = \cos(-\theta)$ this curve is symmetrical about the line $\theta = 0$. As θ increases from 0 to π, r decreases from $2a$ to 0.

$$\frac{dr}{d\theta} = -a\sin\theta$$

so $r\dfrac{d\theta}{dr} = -\dfrac{1+\cos\theta}{\sin\theta} = -\cot\dfrac{1}{2}\theta = \tan\phi$

$\tan\phi = -\cot\dfrac{1}{2}\theta = \tan\dfrac{1}{2}(\pi + \theta)$

so $\phi = \dfrac{1}{2}(\pi + \theta)$

when $\theta = 0, \quad \phi = \frac{1}{2}\pi$ so the curve is at right angles to the initial line.

DIAGRAM

when $\psi = 0$, since $\psi = \phi + \theta$ we have $\theta = -\frac{\pi}{3}$.

when $\theta = \frac{\pi}{3}, \quad \psi = \pi$ so the highest points are at $\theta = \frac{\pi}{3}$.