Question

Find the limit of the sequences:

(a) $u_n = \dfrac{(n+1)^2}{n^2+1}$

(b) $u_n = \dfrac{x^n}{n!}$

Answer

(a) $u_n = \dfrac{(n+1)^2}{n^2+1} = \dfrac{\left(1+\frac{1}{n}\right)^2}{1+\frac{1}{n^2}}$

$$\lim_{n\to\infty} u_n = \frac{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^2}{\lim\limits_{n\to\infty} 1+\dfrac{1}{n^2}} = \frac{1}{1} = 1$$

(b) $\dfrac{u_{n+1}}{u_n} = \dfrac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \dfrac{x}{n+1}$

For $m > n$ $\dfrac{u_{m+1}}{u_m} = \dfrac{x}{m+1} < \dfrac{x}{n+1}$

For any given value of x, $\dfrac{x}{n+1} = r < 1$ provided $n+1 > x$

Hence $u_{n+p} = u_n \times \dfrac{u_{n+1}}{u_n} \times \dfrac{u_{n+2}}{u_{n+1}} \times \ldots \times \dfrac{u_{n+p}}{u_{n+p-1}} < u_n r^p$

If r < 1 $\lim\limits_{p\to\infty} r^p = 0$

Hence $\lim\limits_{p\to\infty} u_{n+p} = 0$

Hence $\lim\limits_{m\to\infty} u_m = 0$