Question

Write down the Taylor expansion for e^x. Then use Maclaurin's theorem to show that if $0 \leq x \leq 1$ then $e^x = 1 + x + \dfrac{x^2}{2} + R(x)$, where $0 \leq R(x) \leq \frac{1}{6}e^x$. Hence show that $1 + x + \dfrac{x^2}{2} \leq e^x \leq \dfrac{6}{5}\left(1 + x + \dfrac{x^2}{2}\right)$. Use your calculator to evaluate e and check these bounds.

Answer

Taylor expansion of e^x around $x = 0$

$e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ...$

Remainder term $R_n(x) = \dfrac{x^n}{n!}e^{\theta x} \quad 0 < \theta < 1$

Hence $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}e^{\theta x}$

e^x is increasing $0 < x < 1 \Rightarrow 0 \leq \dfrac{x^3}{6}e^{\theta x} \leq \dfrac{1}{6}e^x$

Hence $1 + x + \dfrac{x^2}{2!} \leq e^x \leq 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}e^x$

Hence $e^x\left(1 - \dfrac{x^3}{6}\right) \leq 1 + x + \dfrac{x^2}{2}$

$$\begin{aligned}
\Rightarrow e^x &\leq \frac{1 + x + \frac{x^2}{2}}{1 - \frac{x^3}{6}} \\
\text{if } 0 < x < 1, \quad 1 &\geq 1 - \frac{x^3}{6} \geq \frac{5}{6} \\
\text{Hence} \quad 1 &\leq \frac{1}{1 - \frac{x^3}{6}} \leq \frac{6}{5} \\
\text{Hence} \quad e^x &\leq \frac{1 + x + \frac{x^2}{2}}{1 - \frac{x^3}{6}} \leq \frac{6}{5}\left(1 + x + \frac{x^2}{2}\right) \\
\Rightarrow 1 + x + \frac{x^2}{2} &\leq e^x \leq \frac{6}{5}\left(1 + x + \frac{x^2}{2}\right)
\end{aligned}$$

Putting $x = 1$ gives

$$1 + 1 + \frac{1}{2} \leq e \leq \frac{6}{5}\left(1 + 1 + \frac{1}{2}\right),$$

giving $\dfrac{5}{2} \leq e \leq \dfrac{6}{5} \times \dfrac{5}{2}$, i.e. $2.5 \leq e \leq 3$, which is consistent with the value given on the calculator, namely $e = 2.71828\ldots$