Exam Question
Topic: GammaFunction
Using appropriate substitutions, show that

$$\text{(i)} \qquad \int_0^1 \ln\left(\frac{1}{u}\right)^{x-1} du = \Gamma(x),$$

$$\text{(ii)} \qquad \int_0^{\pi/2} \left(\tan^9 x + \tan^{11} x\right) \exp\left(-\tan^2 x\right) dx = 12$$

Solution
(i) Let $u = \mathrm{e}^{-t}$, so $\ln\ln\left(\frac{1}{u}\right) = t$.

$$\int_0^1 \ln\left(\frac{1}{u}\right)^{x-1} du = \int_\infty^0 t^{x-1}(-\mathrm{e}^{-t})\,dt = \int_0^\infty t^{x-1}(-\mathrm{e}^{-t})\,dt = \Gamma(x).$$

$$\begin{aligned}
\text{(ii)} \quad I &= \int_0^{\pi/2} \left(\tan^9 x + \tan^{11} x\right) \exp\left(-\tan^2 x\right) dx \\
&= \frac{1}{2}\int_0^{\pi/2} \left(\tan^2 x\right)^4 .2\tan x(1+\tan^2 x)\exp\left(-\tan^2 x\right) dx \\
&= \frac{1}{2}\int_0^{\pi/2} \left(\tan^2 x\right)^4 .2\tan x\sec^2 x\exp\left(-\tan^2 x\right) dx.
\end{aligned}$$

Let $t = \tan^2 x$; $\quad dt = 2\tan x\sec^2 x\,dx$

$$\text{So} \quad I = \frac{1}{2}\int_0^\infty t^4\mathrm{e}^{-t}\,dt = \frac{\Gamma(5)}{2} = \frac{4!}{2} = 12.$$