Question
A curve is given by $x = a\cos\psi$, $y = a\sin\psi$, $z = \psi$, where $\psi > 0$. Find the following as functions of ψ:

(a) the tangent, principal normal and binormal vectors;

(b) curvature and torsion;

(c) arc length along the curve.

Answer
$\mathbf{r} = a\cos\psi\mathbf{i} + a\sin\psi\mathbf{j} + \psi\mathbf{k}$

(a) Finding the tangent:

$$\begin{aligned}
\frac{d\mathbf{r}}{ds} &= -a\sin\psi\frac{d\psi}{ds}\mathbf{i} + a\cos\psi\frac{d\psi}{ds}\mathbf{j} + \frac{d\psi}{ds}\mathbf{k} \\
\mathbf{t} = \frac{d\mathbf{r}}{ds} &= \frac{d\psi}{ds}[-a\sin\psi\mathbf{i} + a\cos\psi\mathbf{j} + \mathbf{k}] \\
1 = \mathbf{t}\cdot\mathbf{t} &= \left(\frac{d\psi}{ds}\right)^2(1+a^2) \Rightarrow \frac{d\psi}{ds} = \frac{1}{\sqrt{1+a^2}} \\
\mathbf{t} &= \frac{1}{\sqrt{1+a^2}}[-a\sin\psi\mathbf{i} + a\cos\psi\mathbf{j} + \mathbf{k}]
\end{aligned}$$

where we assume s to increase with ψ.

Finding the principal normal:

$$\begin{aligned}
\frac{d\mathbf{t}}{ds} &= \frac{1}{\sqrt{1+a^2}}\frac{d\psi}{ds}[-a\cos\psi\mathbf{i} - a\sin\psi\mathbf{j}] \\
&= \frac{-a}{1+a^2}[\cos\psi\mathbf{i} + \sin\psi\mathbf{j}]
\end{aligned}$$

Now $\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}$ from the Serret-Frenet formulae, so
$\kappa = \frac{a}{1+a^2}$ and $\mathbf{n} = -[\cos\psi\mathbf{i} + \sin\psi\mathbf{j}]$
Finding the binormal vector:

$$\mathbf{m} = \mathbf{t}\times\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\psi & a\cos\psi & 1 \\ -\cos\psi & -sin\psi & 0 \end{vmatrix}\frac{1}{\sqrt{1+a^2}}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{1+a^2}}[\sin\psi\mathbf{i} - \cos\psi\mathbf{j} + a(\sin^2\psi + \cos^2\psi)\mathbf{k}] \\
\mathbf{m} &= \frac{1}{\sqrt{1+a^2}}[\sin\psi\mathbf{i} - \cos\psi\mathbf{j} + a\mathbf{k}]
\end{aligned}
$$

(b) Curvature: $\kappa = \dfrac{a}{1+a^2}$

Torsion: (τ)

$\dfrac{d\mathbf{m}}{ds} = \dfrac{1}{\sqrt{1+a^1}}\dfrac{d\psi}{ds}[\cos\psi\mathbf{i} + \sin\psi\mathbf{j}] = \dfrac{1}{\sqrt{1+a^2}}[\cos\psi\mathbf{i} + \sin\psi\mathbf{j}]$

Now from the Serret-Frenet formulae $\dfrac{d\mathbf{m}}{ds} = -\tau\mathbf{n}$ so $\tau = \dfrac{1}{1+a^2}$

(c) Arc length:

$$
\begin{aligned}
\left(\frac{ds}{d\psi}\right)^2 &= \frac{d\mathbf{r}}{d\psi}\cdot\frac{d\mathbf{r}}{d\psi} = |-a\sin\psi\mathbf{i} + a\cos\psi\mathbf{j} + \mathbf{k}|^2. \\
\text{So } \frac{ds}{d\psi} &= \sqrt{1+a^2} \Rightarrow s = \psi\sqrt{1+a^2}
\end{aligned}
$$

where we have defined $s = 0$ at $\psi = 0$ and s increasing with ψ