Question
Consider the following set of simultaneous equations

$$\begin{aligned} x - y - z &= 0 \\ 3x + y + 2z &= 6 \\ 2x + 2y + kz &= 2 \end{aligned}$$

(a) If $h = 1$, find the solution by matrix inversion.

(b) If $k = 3$, show that the equations do not have a unique solution. (Do NOT attempt to find the solution set.)

Answer

(a) $k = 1$:

$$\begin{aligned} x - y - z &= 0 \\ 3x + y + 2z &= 6 \\ 2x + 2y + z &= 2 \end{aligned}$$

$$\equiv \underset{\mathbf{A}}{\begin{pmatrix} 1 & -1 & -1 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}} \cdot \underset{\mathbf{X}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{\mathbf{K}}{\begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix}}$$

Require $\mathbf{A}^{-1}$, since $\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$

Step (iii)

$$\begin{aligned} \triangle &= detA = \begin{vmatrix} 1 & -1 & -1 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} \\ &= 1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \\ &= 1 \times (1-4) + 1 \times (3-4) - 1 \times (6-2) \\ &= -3 - 1 - 4 \\ &= -8 \end{aligned}$$

Step (i):

Cofactors of the matrix are given by:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

cofactor of $A_{11} = +\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$

cofactor of $A_{12} = -\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = +1$

cofactor of $A_{13} = +\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = +4$

cofactor of $A_{21} = -\begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} = -1$

cofactor of $A_{22} = +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = +3$

cofactor of $A_{23} = -\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = -4$

cofactor of $A_{31} = +\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} = -1$

cofactor of $A_{32} = -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = -5$

cofactor of $A_{33} = +\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = +4$

Matrix of cofactors is thus:

$$\begin{pmatrix} -3 & 1 & 4 \\ -1 & 3 & -4 \\ -1 & -5 & 4 \end{pmatrix}$$

Step (ii):

Transpose to get $adj A$

$$adj\ A = \begin{pmatrix} -3 & -1 & -1 \\ 1 & 3 & -5 \\ 4 & -4 & 4 \end{pmatrix}$$

Step (iii):

$detA = -8$

Step(iv):

$$\begin{aligned}\mathbf{A} &= \frac{adj\ A}{\det\ A} \\ &= -\frac{1}{8}\begin{pmatrix} -3 & -1 & -1 \\ 1 & 3 & -5 \\ 4 & -4 & 4 \end{pmatrix}\end{aligned}$$

Hence

$\mathbf{X} = -\frac{1}{8}\begin{pmatrix} -3 & -1 & -1 \\ 1 & 3 & -5 \\ 4 & -4 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{8}\begin{pmatrix} -8 \\ 8 \\ -16 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(b) when $k = 3$

$$det\begin{pmatrix} 1 & -1 & -1 \\ 3 & 1 & 2 \\ 2 & 2 & k \end{pmatrix} = 0$$

where $k = 3$

Thus no inverse. Hence no unique solution.