Question

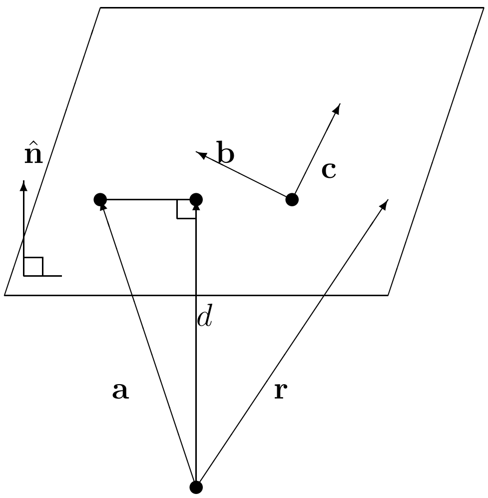
Prove that a vector equation of the plane containing the point $\mathbf{a} = \mathbf{j} + 2\mathbf{k}$ and which contains the vectors $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ is given by $\mathbf{r} \cdot (-3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 3$.

Find the position vector of the point of intersection of this plane with the line

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + \mathbf{k}).$$

What is the angle between this position vector and the normal to the plane?

Answer



$\mathbf{b} and \mathbf{c}$ lie in a plane.

$d = \mathbf{a} \cdot \hat{\mathbf{n}}$ where $\mathbf{n}$ is a unit vector normal to the plane.

A vector normal to the plane is given by $\mathbf{b} \times \mathbf{c}$.

$$\begin{aligned}
\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 1 \\ 3 & 1 & 5 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} -4 & 1 \\ 1 & 5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix} \\
&= -21\mathbf{i} - 7\mathbf{j} + 14\mathbf{k}.
\end{aligned}$$

$$\hat{\mathbf{n}} = \frac{-21\mathbf{i} - 7\mathbf{j} + 14\mathbf{k}}{\sqrt{(-21)^2 + (-7)^2 + (14)^2}}$$

$$
\begin{aligned}
d = \mathbf{k} \cdot \hat{\mathbf{k}} &= (0, +1, 2) \cdot \frac{(-21, -7, 14)}{\sqrt{21^2 + 7^2 + 14^2}} \\
&= \frac{-7 + 28}{\sqrt{21^2 + 7^2 + 14^2}} \\
&= \frac{21}{\sqrt{21^2 + 7^2 + 14^2}}
\end{aligned}
$$

Therefore
$\mathbf{r} \cdot \frac{(-21\mathbf{i} - 7\mathbf{j} + 14\mathbf{k})}{\sqrt{21^2 + 7^2 + 14^2}} = \frac{21}{\sqrt{21^2 + 7^2 + 14^2}}$
$\Rightarrow \mathbf{k} \cdot (-3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 3$

Intersection of plane and line is given by
$[(\mathbf{i}+\mathbf{j}+\mathbf{k})+\lambda(-\mathbf{i}+\mathbf{k})]\cdot(-3\mathbf{i}-\mathbf{j}+2\mathbf{k})=3$
$((1-\lambda),1,1+\lambda)\cdot(-3,-1,2)=3$
$\Rightarrow -3+3\lambda-1+2+2\lambda=3$
$\Rightarrow -2+5\lambda=3$
$\Rightarrow \underline{\lambda=1}$
Therefore

$$\begin{aligned}\mathbf{r}_p &= \mathbf{i}+\mathbf{j}+\mathbf{k}+1(-\mathbf{j}+\mathbf{k})\\ &= \mathbf{j}+2\mathbf{k}\end{aligned}$$

Angle between this and the normal vector is given by,θ where

$$\mathbf{r}_p\cdot\mathbf{n}=|\mathbf{r}_p||\mathbf{n}|\cos\theta$$

$\mathbf{r}_p\cdot\mathbf{n}=(0,1,2)\cdot(-3,-1,2)=3$
$|\mathbf{r}_p|=\sqrt{1^2+2^2}=\sqrt{5}$
$|\mathbf{n}|=\sqrt{3^2+1^2+2^2}=\sqrt{14}$
Therefore $\cos\theta=\dfrac{3}{\sqrt{5}\sqrt{14}}$
$\Rightarrow \theta=\arccos\left(\dfrac{3}{\sqrt{5}\sqrt{14}}\right)=68.98^\circ\approx\underline{69^\circ}$