Question
Solve

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = \sin 3t$$

$$\text{where x} = \frac{-6}{37},\ \frac{\text{dx}}{\text{dt}} = \frac{3}{37},\ \text{when t} = 0$$

Answer
Trial solution is xAe^{kt}.
Solve for complementary function+particular integral.
C.F.: Solution of homogeneous equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0$$

$x = Ae^{kt}$
$\Rightarrow$

$$\begin{aligned} k^2 + 2k + 10 &= 0 \\ k &= \frac{-2 \pm \sqrt{4 - (4 \times 1 \times 10)}}{2} \\ &= -1 \pm \frac{\sqrt{-36}}{2} \\ &= \underline{-1 \pm 3i} \end{aligned}$$

Thus we have a solution:
$x = Ae^{(-1+3i)t} + Be^{(-1-3i)t}$ $A,\ B$ arbitrary
<u>or</u>
$x = e^{-t}(C \cos 3t + D \sin 3t)$ $C,\ D$ arbitrary
P.I.: Should try P.I. of

$$x_{PI} = E \cos 3t + F \sin 3t$$

$\frac{dx}{dt} = -3E \sin 3t + 3F \cos 3t$
$\frac{d^2x}{dt^2} = -9E \cos 3t - 9F \sin 3t$
Substitute into equations;
$-9E \cos 3t - 9F \sin 3t + 2(-3E \sin 3t + 3F \cos 3t)$
$+10E \cos 3t + 10F \sin 3t = \sin 3t$
Balance of $\cos 3t$ and $\sin 3t$
$-9E + 6F + 10E = 0 \quad (\cos 3t)$

$-9F - 6E + 10F = 1 \quad (\sin 3t)$
$E + 6F = 0 \quad (1)$
$-6E + F = 1 \quad (2)$
$(1) - 6 \times (2)$:

$$\begin{array}{l} E + 6F = 0 \\ \underline{-36E + 6F = 6} \\ 37E \qquad = -6 \end{array}$$

$$E = -\frac{6}{37}, \; F = \frac{1}{37}$$

Therefore $x = x_{CF} + x_{PI} = e^{-t}(C\cos 3t + D\sin 3t) + \frac{1}{37}\sin 3t - \frac{6}{37}\cos 3t$
Boundary conditions:
(1) $x = 1$ when $t = 0$:
$\Rightarrow -\frac{6}{37} = e^0(C\cos 0 + 0) + 0 - \frac{6}{37}\cos 0$
$\Rightarrow -\frac{6}{37} = C - \frac{6}{37}$
$\Rightarrow C = \underline{0}$
Therefore $x = De^{-t}\sin 3t + \frac{1}{37}\sin 3t - \frac{6}{37}\cos 3t$
(2) $\frac{dx}{dt} = D(-e^{-t}\sin 3t + 3e^{-t}\cos 3t) + \frac{3}{37}\cos 3t + \frac{18}{37}\sin 3t$
Hence if $\frac{dx}{dt} = \frac{3}{37}$ when $t = 0$
$\frac{3}{37} = D(0 + 3) + \frac{3}{37} + 0$
$\Rightarrow \underline{D = 0}$
Thus total solution is: (CF vanishes!)

$$\underline{x = \frac{1}{37}\sin 3t - \frac{6}{37}\cos 3t}$$