Question

(a) Solve the differential equation

$$\frac{dy}{dx} + 2xy = e^{-x^2}\sec^2 x, \text{ where } y = 1 \text{ when } x = 0$$

(b) By solving

$$\frac{dy}{dx} = \frac{3x+2y}{2x+3y}$$

show that its general solution can be written in the form

$(x-y)^5(x+y) = A$

where A is an arbitrary constant.

Answer

(i) $\dfrac{dy}{dx} + 2xy = e^{-x^2}\sec^2 x$

First order linear equation, requiring an integrating factor:

$\dfrac{dy}{dx} + Py = Q \Rightarrow$ integrating factor $e^{\int P\,dx}$

Here $P = 2x,\ Q = e^{-x^2}\sec^2 x$

This integrating factor$=e^{\int 2x\,dx} = e^{x^2}$

Therefore

$$\begin{aligned}
e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y &= e^{x^2-x^2}\sec^2 x \\
e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y &= \sec^2 x \\
\frac{d}{dx}\left(e^{x^2}y\right) &= \sec^2 x \\
\Rightarrow ye^{x^2} &= \int^x \sec^2 u\,du \\
&= \tan x + C \\
\Rightarrow y &= \underline{e^{-x^2}\tan x + Ce^{-x^2}}
\end{aligned}$$

General solution.

If $y = 1$ when $x = 0$

$\Rightarrow y = e^0 \cdot 0 + Ce^0 = C$

$\Rightarrow C = 1$

$$\underline{y = (1 + \tan x)e^{-x^2}}$$

Specific solution.

(ii) $\dfrac{dy}{dx} = \dfrac{3x + 2y}{2x + 3y} \quad (1)$

Rearrange to get

$(2x + 3y)\dfrac{dy}{dx} - (3x + 2y) = 0$

Homogeneous of degree $\underline{1}$.

Set $y = vx$ where $v = v(x)$, to be found.

$\Rightarrow = v + x\dfrac{dv}{dx}$

Substitute in (1)

$$\begin{aligned}
\Rightarrow v + x\frac{dv}{dx} &= \frac{3x + 2vx}{2x + 3vx} \\
\Rightarrow v\frac{dv}{dx} &= \frac{3 + 2v}{2 + 3v} \\
x\frac{dv}{dx} &= \frac{3 + 2v}{2 + 3v} - v \\
x\frac{dv}{dx} &= \frac{3 + 2v - 2v - 3v^2}{2 + 3v} \\
x\frac{dv}{dx} &= \frac{3(1 - v^2)}{2 + 3v}
\end{aligned}$$

Variables separable:

$$\int \frac{(2+3v)}{(1-v^2)}\,dv = 3\int \frac{dx}{x}$$

$\displaystyle\int \frac{2+3v}{1-v^2}\,dv = 3\ln x + C$

$\displaystyle\frac{2+3v}{1-v^2} = \frac{(2+3v)}{(1-v)(1+v)} \equiv \frac{A}{(1-v)} + \frac{B}{(1+v)}$

Therefore $2+3v = (1+v)A + (1-v)B$

$\Rightarrow 2 = A+B,\ 3 = A-B$

$\Rightarrow 5 = 2A \Rightarrow A = \dfrac{5}{2}$

Hence $B = -\dfrac{1}{P}2$

Therefore

$$\begin{aligned}\int \frac{2+3v}{1-v^2}\,dv &= \frac{5}{2}\int \frac{dv}{1-v} - \frac{1}{2}\int \frac{dv}{1+v}\\ &= -\frac{5}{2}\ln(1-v) - \frac{1}{2}\ln(1+v)\end{aligned}$$

Therefore

$$\begin{aligned}-\frac{5}{2}\ln(1-v) - \frac{1}{2}\ln(1+v) &= 3\ln x + C\\ &= -\frac{5}{2}\ln(1-v) - \frac{1}{2}\ln(1+v)\end{aligned}$$

Therefore

$$\begin{aligned}-\frac{5}{2}\ln(1-v) - \frac{1}{2}\ln(1+v) &= 3\ln x + c\\ 5\ln(1-v) + \ln(1+v) &= -6\ln x + c'\\ (1-v)^5(1+v) &= \frac{A}{x^6}\\ \text{reset y = vx}\\ \left(1-\frac{y}{x}\right)^5\left(1+\frac{y}{x}\right) &= \frac{A}{x^6}\\ \Rightarrow (x-5)^5(x+y) &= A\end{aligned}$$