Question

Show that the centre of mass of a right circular cone of radius r and height h is situated at a distance $\frac{h}{4}$ from the base.

A missile is to be constructed from two parts. The propulsion unit is effectively a cylinder of radius r, height H and density $\frac{\omega}{2}$. The warhead is situated in the nose cone, which is a right circular cone of radius r, height h and density ω.

Show that the centre of mass of the missile is at a distance

$$\frac{(H^2+4Hh+h^2)}{2(3H+2h)}$$

from the base of the propulsion unit.

Hence show that if $H = h = h\frac{5r}{4}$ and the missile is placed on a perfectly rough slope inclined at an angle of less than $\frac{\pi}{4}$ between its axis of symmetry and the horizontal, it is liable to topple over.

Answer

PICTURE

Density=ω

By symmetry centre of mass is along axis of symmetry of cone at $(\bar{x}, 0)$

Elemental disc: mass= $\pi r(x)^2\omega\partial x$

By similar triangles $\frac{r}{h} = \frac{r(x)}{x}$

So mass= $\pi x^2\frac{r^2}{h^2}\omega\partial x$

moment of disc about y axis= $\underline{\pi\frac{x^3r^2}{h^2}\omega\partial x}$

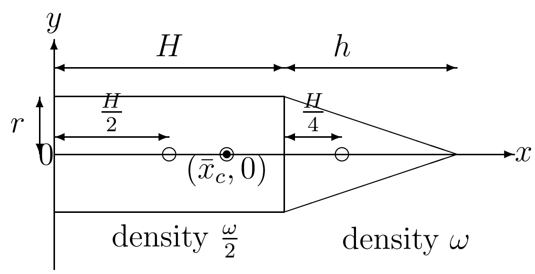
Balance total moments:

$$M\bar{x} = \int_0^h \frac{\pi x^3r^2\omega}{h^2}\partial x$$

where M =mass of come = $\frac{\pi r^2\omega h}{3}$

so $\bar{x} = \frac{\frac{\pi r^2}{h^2}\omega[\frac{x^4}{4}]_0^h}{\frac{\pi r^2\omega h}{3}}$

$$\underline{\bar{x} = \frac{3h}{4}; \quad \text{i.e.,} \frac{\text{h}}{4} \text{ from base}}$$



Let centre of mass of composite body be at $(\bar{x}_c, 0)$ (by symmetry).
Take moments about y axis:

1. moment cylinder$=\underbrace{\pi r^2 H \frac{\omega}{2}}_{\text{mass}} \times \underbrace{\frac{H}{2}}_{\text{c of m}}$
2. moment warhead$=\frac{1}{3}\pi r^2 h\omega \times \left(H + \frac{h}{4}\right)$
3. moment missile$=\left(\underbrace{\frac{1}{3}\pi r^2 h\omega}_{\text{mass warhead}} + \underbrace{\frac{\pi r^2 H\omega}{2}}_{\text{mass cylinder}}\right)\bar{x}_c$

Now (3) = (1) + (2)

$$\begin{aligned}
\Rightarrow \left(\frac{1}{3}\pi r^2 h\omega + \frac{\pi r^2 H\omega}{2}\right)\bar{x}_c &= \frac{\pi r^2 H^2\omega}{4} + \frac{\pi r^2 h\omega}{3}\left(H + \frac{h}{4}\right) \\
\Rightarrow \bar{x}_c &= \frac{\pi r^2\omega(\frac{H^2}{4} + \frac{hH}{3} + \frac{h^2}{12})}{\pi r^2\omega(\frac{h}{3} + \frac{H}{2})} \\
&= \underline{\frac{3H^2 + 4Hh + h^2}{2(2h + 3H)}}
\end{aligned}$$

as required.
Take $H = h = \frac{5r}{4} \Rightarrow \bar{x}_c = \frac{3h^2 + 4h^2 + h^2}{2(2h + 3h)} = \frac{4h}{5} = r$
PICTURE

Assuming centre of mass $\approx$ centre of gravity, we have that the critical point of toppling occurs when centre of mass lies just outside base of missile,
i.e., $\tan\theta = \frac{r}{r} = 1 \Rightarrow \underline{\theta = \frac{\pi}{4}}$
Any angle greater than this and the missile topples over.