Question

Prove, if $\{a_n\}$ is a sequence satisfying $\lim_{n\to\infty} |a_n|^{1/n} = L \neq 0$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $\frac{1}{L}$.

Answer

The condition that the a_n satisfy is similar to the condition of the root test, and so we apply the root test to the power series $\sum_{n=0}^{\infty} a_n x^n$. Namely, we calculate

$$\lim_{n\to\infty} |a_n x^n|^{1/n} = |x| \lim_{n\to\infty} |a_n|^{1/n} = L|x|.$$

Hence, the series converges absolutely for $L|x| < 1$, that is $|x| < \frac{1}{L}$, and diverges for $L|x| > 1$, and so the radius of convergence of this series is $\frac{1}{L}$, as desired.