Question

The power series scavenger hunt: for each of the power series given below, determine the radius and interval of convergence.

1. $\sum_{n=0}^{\infty}(-1)^n x^n/n!$;
2. $\sum_{n=1}^{\infty} 5^n x^n/n^2$;
3. $\sum_{n=1}^{\infty} x^n/(n(n+1))$;
4. $\sum_{n=1}^{\infty}(-1)^n x^n/\sqrt{n}$;
5. $\sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$;
6. $\sum_{n=0}^{\infty} 3^n x^n/n!$;
7. $\sum_{n=0}^{\infty} x^n/(1+n^2)$;
8. $\sum_{n=1}^{\infty}(-1)^{n+1}(x+1)^n/n$;
9. $\sum_{n=0}^{\infty} 3^n(x+5)^n/4^n$;
10. $\sum_{n=1}^{\infty}(-1)^n(x+1)^{2n+1}/(n^2+4)$;
11. $\sum_{n=0}^{\infty} \pi^n(x-1)^{2n}/(2n+1)!$;
12. $\sum_{n=2}^{\infty} x^n/(\ln(n))^n$;
13. $\sum_{n=0}^{\infty} 3^n x^n$;
14. $\sum_{n=0}^{\infty} n!x^n/2^n$;
15. $\sum_{n=1}^{\infty}(-2)^n x^{n+1}/(n+1)$;
16. $\sum_{n=1}^{\infty}(-1)^n x^{2n}/(2n)!$;
17. $\sum_{n=1}^{\infty}(-1)^n x^{3n}/n^{3/2}$;
18. $\sum_{n=2}^{\infty}(-1)^{n+1}x^n/(n\ln^2(n))$;
19. $\sum_{n=0}^{\infty}(x-3)^n/2^n$;
20. $\sum_{n=1}^{\infty}(-1)^n(x-4)^n/(n+1)^2$;
21. $\sum_{n=0}^{\infty}(2n+1)!(x-2)^n/n^3$;
22. $\sum_{n=1}^{\infty} \ln(n)(x-3)^n/n$;

23. $\sum_{n=0}^{\infty}(2x-3)^n/4^{2n}$;

24. $\sum_{n=2}^{\infty}(x-a)^n/b^n$, where $b>0$ is arbitrary.

25. $\sum_{n=0}^{\infty}(n+p)!x^n/(n!(n+q)!)$, where p, $q\in\mathbf{N}$;

26. $\sum_{n=1}^{\infty} x^{n-1}/(n3^n)$;

27. $\sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-1}/(2n-1)!$;

28. $\sum_{n=1}^{\infty} n!(x-a)^n$, where $a\in\mathbf{R}$ is arbitrary;

29. $\sum_{n=1}^{\infty} n(x-1)^n/(2^n(3n-1))$;

Answer

1. **radius of convergence is ∞, interval of convergence is $\mathbf{R}$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}/(n+1)!}{(-1)^n x^n/n!}\right| = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Hence, this series converges absolutely for all values of x (since this limit is 0 for every value of x).

2. **radius of convergence is $\frac{1}{5}$, interval of convergence is $\left[-\frac{1}{5},\frac{1}{5}\right]$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{5^{n+1}x^{n+1}/(n+1)^2}{5^n x^n/n^2}\right| = |x|\lim_{n\to\infty}\frac{5n^2}{(n+1)^2} = 5|x|.$$

Hence, this series converges absolutely for $5|x|<1$, that is for $|x|<\frac{1}{5}$, and so the radius of convergence is $\frac{1}{5}$. We now need to check the endpoints of the interval $(-\frac{1}{5},\frac{1}{5})$:

At $x=-\frac{1}{5}$, the series becomes $\sum_{n=1}^{\infty} 5^n(-1/5)^n/n^2 = \sum_{n=1}^{\infty}(-1)^n/n^2$, which converges absolutely.

At $x=\frac{1}{5}$, the series becomes $\sum_{n=1}^{\infty} 5^n(1/5)^n/n^2 = \sum_{n=1}^{\infty} 1/n^2$, which converges absolutely.

So the series converges absolutely for all x in the closed interval $\left[-\frac{1}{5},\frac{1}{5}\right]$, and diverges elsewhere.

3. **radius of convergence is 1, interval of convergence is $[-1,1]$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{x^{n+1}/((n+1)(n+2))}{x^n/(n(n+1))}\right| = |x|\lim_{n\to\infty}\frac{n}{n+2} = |x|.$$

Hence, this series converges absolutely for $|x| < 1$, and so the radius of convergence is 1. We now need to check the endpoints of the interval $(-1,1)$:

At $x=-1$, the series becomes $\sum_{n=1}^{\infty}(-1)^n/(n(n+1))$, which converges absolutely.

At $x=1$, the series becomes $\sum_{n=1}^{\infty}1/(n(n+1))$, which converges absolutely.

So, the series converges absolutely for all x in the closed interval $[-1,1]$, and diverges elsewhere.

4. **radius of convergence is 1, interval of convergence is $[-1,1)$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}/\sqrt{n+1}}{(-1)^n x^n/\sqrt{n}}\right| = |x|\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = |x|.$$

Hence, this series converges absolutely for $|x| < 1$, and so the radius of convergence is 1. We now need to check the endpoints of the interval $(-1,1)$:

At $x=-1$, the series becomes $\sum_{n=1}^{\infty}(-1)^n/\sqrt{n}$, which converges conditionally. (The alternating series test yields convergence, but this series does not converge absolutely, by comparison to the harmonic series.)

At $x=1$, the series becomes $\sum_{n=1}^{\infty}1/\sqrt{n}$, which diverges.

So, the series converges absolutely for all x in the open interval $(-1,1)$, converges conditionally at $x=-1$, and diverges elsewhere.

5. **radius of convergence is ∞, interval of convergence is R:** Apply the ratio test and calculate:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)+1}/(2(n+1)+1)!}{(-1)^n x^{2n+1}/(2n+1)!}\right| &= |x|^2\lim_{n\to\infty}\frac{(2n+1)!}{(2n+3)!}\\ &= |x|^2\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)} = 0.\end{aligned}$$

Hence, this series converges absolutely for all values of x (since this limit is 0 for every value of x).

6. **radius of convergence is ∞, interval of convergence is R:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{3^{n+1}x^{n+1}/(n+1)!}{3^n x^n/n!}\right| = |x|\lim_{n\to\infty}\frac{3}{n+1} = 0.$$

Hence, this series converges absolutely for all values of x (since this limit is 0 for every value of x).

7. **radius of convergence is 1, interval of convergence is $[-1,1]$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{x^{n+1}/(1+(n+1)^2)}{x^n/(1+n^2)}\right| = |x|\lim_{n\to\infty}\frac{1+n^2}{2+2n+n^2} = |x|.$$

Hence, this series converges absolutely for $|x|<1$, and so the radius of convergence is 1. We now need to check the endpoints of the interval $(-1,1)$:

At $x=-1$, the series becomes $\sum_{n=0}^{\infty}(-1)^n/(1+n^2)$, which converges absolutely.

At $x=1$, the series becomes $\sum_{n=0}^{\infty}1/(1+n^2)$, which converges absolutely.

So, the series converges absolutely for all x in the closed interval $[-1,1]$, and diverges elsewhere.

8. **radius of convergence is 1, interval of convergence is $(-2,0]$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(-1)^{(n+1)+1}(x+1)^{n+1}/(n+1)}{(-1)^{n+1}(x+1)^n/n}\right| = |x+1|\lim_{n\to\infty}\frac{n}{n+1} = |x+1|.$$

Hence, this series converges absolutely for $|x+1|<1$, and so the radius of convergence is 1. We now need to check the endpoints of the interval $(-2,0)$:

At $x=-2$, the series becomes $\sum_{n=1}^{\infty}(-1)^{n+1}(-1)^n/n = -\sum_{n=1}^{\infty}1/n$, which diverges, being a constant multiple of the harmonic series.

At $x = 0$, the series becomes $\sum_{n=1}^{\infty}(-1)^{n+1}/n$, which converges conditionally, as it is the alternating harmonic series.

So, the series converges absolutely for all x in the open interval $(-2, 0)$, converges conditionally at $x = 0$, and diverges elsewhere.

9. **radius of convergence is $\frac{4}{3}$, interval of convergence is $(-\frac{19}{3}, -\frac{11}{3})$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{3^{n+1}(x+5)^{n+1}/4^{n+1}}{3^n(x+5)^n/4^n}\right| = \frac{3}{4}|x+5|.$$

Hence, this series converges absolutely for $\frac{3}{4}|x+5| < 1$, that is for $|x+5| < \frac{4}{3}$, and so the radius of convergence is $\frac{4}{3}$. We now need to check the endpoints of the interval $(-\frac{19}{3}, -\frac{11}{3})$.

At $x = -\frac{19}{3}$, the series becomes

$$\sum_{n=0}^{\infty}\frac{3^n\left(-\frac{19}{3}+5\right)^n}{4^n} = \sum_{n=0}^{\infty}(-1)^n,$$

which diverges (being, for instance, a divergent geometric series).

At $x = -\frac{11}{3}$, the series becomes

$$\sum_{n=0}^{\infty}\frac{3^n\left(-\frac{11}{3}+5\right)^n}{4^n} = \sum_{n=0}^{\infty}1,$$

which diverges (again being, for instance, a divergent geometric series).

So, the series converges absolutely for all x in the open interval $(-\frac{19}{3}, -\frac{11}{3})$, and diverges elsewhere.

10. **radius of convergence is 1, interval of convergence is $[-2, 0]$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x+1)^{2(n+1)+1}/((n+1)^2+4)}{(-1)^n(x+1)^{2n+1}/(n^2+4)}\right| = |x+1|^2\lim_{n\to\infty}\frac{n^2+4}{n^2+2n+5} = |x+1|^2.$$

Hence, this series converges absolutely for $|x+1|^2 < 1$, that is for $|x+1| < 1$, and so the radius of convergence is 1. We now need to check the endpoints of the interval $(-2, 0)$.

At $x=-2$, the series becomes $\sum_{n=1}^{\infty}(-1)^n(-1)^{2n+1}/(n^2+4) = \sum_{n=1}^{\infty}(-1)^{n+1}/(n^2+4)$, which converges absolutely.

At $x=0$, the series becomes $\sum_{n=1}^{\infty}(-1)^n/(n^2+4)$, again which converges absolutely.

So, the series converges absolutely for all x in the closed interval $[-2,0]$, and diverges elsewhere.

11. **radius of convergence is ∞, interval of convergence is R:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{\pi^{n+1}(x-1)^{2(n+1)}/(2(n+1)+1)!}{\pi^n(x-1)^{2n}/(2n+1)!}\right| = |x-1|^2\lim_{n\to\infty}\frac{\pi}{(2n+2)(2n+3)} = 0.$$

Hence, this series converges absolutely for all values of x (since this limit is 0 for every value of x).

12. **radius of convergence is ∞, interval of convergence is R:** This time, since the coefficients are n^{th} powers, we apply the root test and calculate:

$$\lim_{n\to\infty}\left|\frac{x^n}{(\ln(n))^n}\right|^{1/n} = |x|\lim_{n\to\infty}\frac{1}{\ln(n)} = 0.$$

Hence, this series converges absolutely for all values of x (since this limit is 0 for every value of x).

13. **radius of convergence is $\frac{1}{3}$, interval of convergence is $(-\frac{1}{3},\frac{1}{3})$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{3^{n+1}x^{n+1}}{3^n x^n}\right| = 3|x|.$$

Hence, this series converges absolutely for $3|x|<1$, that is $|x|<\frac{1}{3}$, and so the radius of convergence is $\frac{1}{3}$. We now need to check the endpoints of the interval $(-\frac{1}{3},\frac{1}{3})$.

At $x=-\frac{1}{3}$, the series becomes

$$\sum_{n=0}^{\infty}3^n\left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n,$$

which diverges (being, for instance, a divergent geometric series).

At $x = \frac{1}{3}$, the series becomes

$$\sum_{n=0}^{\infty} 3^n \left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} 1,$$

which diverges (again being, for instance, a divergent geometric series).

So, the series converges absolutely for all x in the open interval $(-\frac{1}{3}, \frac{1}{3})$, and diverges elsewhere.

14. **radius of convergence is 0, interval of convergence is $\{0\}$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}/2^{n+1}}{n!x^n/2^n} \right| = |x| \lim_{n\to\infty} \frac{n+1}{2} = \infty.$$

Hence, this series converges only for $x = 0$ and diverges elsewhere.

15. **radius of convergence is $\frac{1}{2}$, interval of convergence is $(-\frac{1}{2}, \frac{1}{2}]$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1}x^{(n+1)+1}/((n+1)+1)}{(-2)^n x^{n+1}/(n+1)} \right| = |x| \frac{2(n+1)}{n+2} = 2|x|.$$

Hence, this series converges absolutely for $2|x| < 1$, that is $|x| < \frac{1}{2}$, and so the radius of convergence is $\frac{1}{2}$. We now need to check the endpoints of the interval $(-\frac{1}{2}, \frac{1}{2})$.

At $x = -\frac{1}{2}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-2)^n \left(-\frac{1}{2}\right)^{n+1}}{n+1} = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n+1},$$

which diverges, as it is a constant multiple of the harmonic series.

At $x = \frac{1}{2}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-2)^n \left(\frac{1}{2}\right)^{n+1}}{n+1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1},$$

which converges, as it is a constant multiple of the alternating harmonic series.

So, the series converges absolutely for all x in the open interval $(-\frac{1}{2}, \frac{1}{2})$, converges conditionally at $x = \frac{1}{2}$, and diverges elsewhere.

16. **radius of convergence is ∞, interval of convergence is R:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}/(2(n+1))!}{(-1)^n x^{2n}/(2n)!}\right| = |x|^2 \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0.$$

Hence, this series converges absolutely for all values of x (since this limit is 0 for every value of x).

17. **radius of convergence is 1, interval of convergence is $[-1,1]$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{3(n+1)}/(n+1)^{3/2}}{(-1)^n x^{3n}/n^{3/2}}\right| = |x|^3 \lim_{n\to\infty}\frac{n^{3/2}}{(n+1)^{3/2}} = |x|^3.$$

Hence, this series converges absolutely for $|x|^3 < 1$, that is $|x| < 1$, and so the radius of convergence is 1. We now need to check the endpoints of the interval $(-1,1)$.

At $x = -1$, the series becomes $\sum_{n=1}^{\infty}(-1)^n(-1)^n/n^{3/2} = \sum_{n=1}^{\infty} 1/n^{3/2}$, which converges, by Note 1.

At $x = 1$, the series becomes $\sum_{n=1}^{\infty}(-1)^n/n^{3/2}$, which converges absolutely, by Note 1.

So, the series converges absolutely for all x in the closed interval $[-1,1]$, and diverges elsewhere.

18. **radius of convergence is 1, interval of convergence is $[-1,1]$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(-1)^{(n+1)+1}x^{n+1}/((n+1)\ln^2(n+1))}{(-1)^{n+1}x^n/(n\ln^2(n))}\right| = |x| \lim_{n\to\infty}\frac{n\ln^2(n)}{(n+1)\ln^2(n+1)} = |x|.$$

Hence, this series converges absolutely for $|x| < 1$, and so the radius of convergence is 1. We now need to check the endpoints of the interval $(-1,1)$.

At $x = -1$, the series becomes $\sum_{n=2}^{\infty}(-1)^{n+1}(-1)^n/(n\ln^2(n)) = -\sum_{n=2}^{\infty} 1/(n\ln^2(n))$, which converges by the integral test: take $f(x) = 1/(x\ln^2(x))$. Then,

$$f'(x) = \frac{-(\ln^2(x) + 2\ln(x))}{x^2\ln^4(x)} < 0$$

for $x \geq 2$, and so $f(x)$ is decreasing. Then, we evaluate

$$\begin{aligned}
\int_2^\infty f(x)\mathrm{d}x &= \lim_{M\to\infty} \int_2^M \frac{1}{x\ln^2(x)}\mathrm{d}x \\
&= \lim_{M\to\infty} \frac{-1}{\ln(x)}\Big|_2^M \\
&= \lim_{M\to\infty} \left(\frac{-1}{\ln(M)} + \frac{1}{\ln(2)}\right) = \frac{1}{\ln(2)},
\end{aligned}$$

which converges. Hence, by the integral test, the series converges.

At $x = 1$, the series becomes $\sum_{n=2}^\infty (-1)^{n+1}/(n\ln^2(n))$, which converges absolutely by the argument just given.

So, the series converges absolutely for all x in the closed interval $[-1, 1]$, and diverges elsewhere.

19. **radius of convergence is 2, interval of convergence is $(1, 5)$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty} \left|\frac{(x-3)^{n+1}/2^{n+1}}{(x-3)^n/2^n}\right| = \frac{1}{2}|x-3|.$$

Hence, this series converges absolutely for $\frac{1}{2}|x-3| < 1$, that is $|x-3| < 2$, and so the radius of convergence is 2. We now need to check the endpoints of the interval $(1, 5)$.

At $x = 1$, the series becomes $\sum_{n=0}^\infty (-2)^n/2^n = \sum_{n=0}^\infty (-1)^n$, which diverges, being for instance a divergent geometric series.

At $x = 5$, the series becomes $\sum_{n=0}^\infty 1$, which diverges, again being for instance a divergent geometric series.

So, the series converges absolutely for all x in the open interval $(1, 5)$, and diverges elsewhere.

20. **radius of convergence is 1, interval of convergence is $[3, 5]$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x-4)^{n+1}/((n+1)+1)^2}{(-1)^n(x-4)^n/(n+1)^2}\right| = |x-4| \lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2} = |x-4|.$$

Hence, this series converges absolutely for $|x-4| < 1$, and so the radius of convergence is 1. We now need to check the endpoints of the interval $(3, 5)$.

At $x = 3$, the series becomes $\sum_{n=1}^{\infty}(-1)^n(-1)^n/(n+1)^2 = \sum_{n=1}^{\infty} 1/(n+1)^2$, which converges by Note 1.

At $x = 5$, the series becomes $\sum_{n=1}^{\infty}(-1)^n/(n+1)^2$, which converges absolutely, again by Note 1.

So, the series converges absolutely for all x in the closed interval $[3, 5]$, and diverges elsewhere.

21. **radius of convergence is 0, interval of convergence is $\{2\}$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(2(n+1)+1)!\,(x-2)^{n+1}/(n+1)^3}{(2n+1)!\,(x-2)^n/n^3}\right| = |x-2|\lim_{n\to\infty}\frac{(2n+3)!\,n^3}{(2n+1)!\,(n+1)^3} = \infty$$

for all $x \neq 2$. Hence, the series converges only for $x = 2$.

22. **radius of convergence is 1, interval of convergence is $[2, 4)$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{\ln(n+1)(x-3)^{n+1}/(n+1)}{\ln(n)(x-3)^n/n}\right| = |x-3|\frac{n\ln(n+1)}{(n+1)\ln(n)} = |x-3|.$$

Hence, this series converges absolutely for $|x-3| < 1$, and so the radius of convergence is 1. We now need to check the endpoints of the interval $(2, 4)$.

At $x = 2$, the series becomes $\sum_{n=1}^{\infty}\ln(n)(-1)^n/n$, which converges by the alternating series test (but does not converge absolutely).

At $x = 4$, the series becomes $\sum_{n=1}^{\infty}\ln(n)/n$, which diverges by the first comparison test, since $\ln(n)/n > 1/n$ for $n \geq 3$ and the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges.

So, the series converges absolutely for all x in the open interval $(2, 4)$, converges conditionally at $x = 2$, and diverges elsewhere.

23. **radius of convergence is 8, interval of convergence is $(-\frac{13}{2}, \frac{19}{2})$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(2x-3)^{n+1}/4^{2(n+1)}}{(2x-3)^n/4^{2n}}\right| = \frac{1}{16}\,|2x-3| = \frac{1}{8}\left|x-\frac{3}{2}\right|.$$

Hence, this series converges absolutely for $\frac{1}{8}\left|x-\frac{3}{2}\right| < 1$, that is for $\left|x-\frac{3}{2}\right| < 8$, and so the radius of convergence is 8. We now need to check the endpoints of the interval $(-\frac{13}{2}, \frac{19}{2})$.

At $x = -\frac{13}{2}$, the series becomes $\sum_{n=0}^{\infty}(2(-13/2)-3)^n/4^{2n} = \sum_{n=0}^{\infty}(-1)^n$, which diverges.

At $x = \frac{19}{2}$, the series becomes $\sum_{n=0}^{\infty}(2(19/2)-3)^n/4^{2n} = \sum_{n=0}^{\infty} 1$, which diverges.

So, the series converges absolutely for all x in the open interval $(-\frac{13}{2}, \frac{19}{2})$, and diverges elsewhere.

24. **radius of convergence is b, interval of convergence is $(a-b, a+b)$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(x-a)^{n+1}/b^{n+1}}{(x-a)^n/b^n}\right| = \frac{1}{b}|x-a|.$$

Hence, this series converges absolutely for $\frac{1}{b}|x-a| < 1$, that is for $|x-a| < b$, and so the radius of convergence is b. We now need to check the endpoints of the interval $(a-b, a+b)$.

At $x = a-b$, the series becomes $\sum_{n=2}^{\infty}(a-b-a)^n/b^n = \sum_{n=2}^{\infty}(-1)^n$, which diverges.

At $x = a+b$, the series becomes $\sum_{n=2}^{\infty}(a+b-a)^n/b^n = \sum_{n=2}^{\infty} 1$, which diverges.

So, the series converges absolutely for all x in the open interval $(a-b, a+b)$, and diverges elsewhere. (Note that the previous series is a specific example of this general phenomenon, with $a = \frac{3}{2}$ and $b = 8$.)

25. **radius of convergence is ∞, interval of convergence is R:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{((n+1)+p)!x^{n+1}/((n+1)!((n+1)+q)!)}{(n+p)!x^n/(n!(n+q)!)}\right| = |x|\lim_{n\to\infty}\frac{n+1+p}{(n+1)(n+1+q)} = 0.$$

Hence, this series converges absolutely for all values of x (since this limit is 0 for every value of x).

26. **radius of convergence is 3, interval of convergence is $[-3, 3)$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{x^{(n+1)-1}/((n+1)3^{n+1})}{x^{n-1}/(n3^n)}\right| = |x|\lim_{n\to\infty}\frac{n}{3(n+1)} = \frac{1}{3}|x|.$$

Hence, this series converges absolutely for $\frac{1}{3}|x| < 1$, that is for $|x| < 3$, and so the radius of convergence is 3. We now need to check the endpoints of the interval $(-3, 3)$.

At $x = -3$, the series becomes $\sum_{n=1}^{\infty}(-3)^{n-1}/(n3^n) = \frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$, which converges conditionally, as it is a constant multiple of the alternating harmonic series.

At $x = 3$, the series becomes $\sum_{n=1}^{\infty} 3^{n-1}/(n3^n) = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges, as it is a constant multiple of the harmonic series.

So, the series converges absolutely for all x in the open interval $(-3, 3)$, converges conditionally at $x = -3$, and diverges elsewhere.

27. **radius of convergence is ∞, interval of convergence is R:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(-1)^{(n+1)-1}x^{2(n+1)-1}/(2(n+1)-1)!}{(-1)^{n-1}x^{2n-1}/(2n-1)!}\right| = |x|^2 \lim_{n\to\infty}\frac{1}{2n(2n+1)} = 0.$$

Hence, this series converges absolutely for all values of x (since this limit is 0 for every value of x).

28. **radius of convergence is 0, interval of convergence is $\{a\}$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(n+1)!(x-a)^{n+1}}{n!(x-a)^n}\right| = |x-a| \lim_{n\to\infty}(n+1) = \infty$$

for all $x \neq a$. Hence, the series converges only for $x = a$.

29. **radius of convergence is 2, interval of convergence is $(-1, 3)$:** Apply the ratio test and calculate:

$$\lim_{n\to\infty}\left|\frac{(n+1)(x-1)^{n+1}/(2^{n+1}(3(n+1)-1))}{n(x-1)^n/(2^n(3n-1))}\right| = |x-1| \lim_{n\to\infty}\frac{(n+1)(3n-1)}{2n(3n+2)} = \frac{1}{2}|x-1|.$$

Hence, this series converges absolutely for $\frac{1}{2}|x-1| < 1$, that is for $|x-1| < 2$, and so the radius of convergence is 2. We now need to check the endpoints of the interval $(-1, 3)$.

At $x = -1$, the series becomes $\sum_{n=1}^{\infty} n(-1-1)^n/(2^n(3n-1)) = \sum_{n=1}^{\infty}(-1)^n n/(3n-1)$, which diverges by the n^{th} term test for divergence, as $\lim_{n\to\infty}\frac{n}{3n-1} = \frac{1}{3}$, and so $\lim_{n\to\infty}\frac{(-1)^n n}{3n-1}$ does not exist.

At $x = 3$, the series becomes $\sum_{n=1}^{\infty} n(3-1)^n/(2^n(3n-1)) = \sum_{n=1}^{\infty} n/(3n-1)$, which again diverges by the n^{th} term test for divergence.

So, the series converges absolutely for all x in the open interval $(-1, 3)$, and diverges elsewhere.

Note 1.
The series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges if and only if $s > 1$.

For $s = 1$, this series is called the **harmonic series**, and we can prove directly that it diverges. Note that $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$, that $\frac{1}{5} + \cdots + \frac{1}{8} > 4\frac{1}{8} = \frac{1}{2}$, and in general that

$$\frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k} > 2^{k-1}\frac{1}{2^k} = \frac{1}{2}.$$

Hence, the $(2^k)^{th}$ partial sum S_{2^k} satisfies $S_{2^k} > 1 + k\frac{1}{2}$. Since the terms in the harmonic series are all positive, the sequence of partial sums is monotonically increasing, and by the calculation done the sequence of partial sums is unbounded, and so the sequence of partial sums diverges. Hence, the harmonic series diverges.