Question
Consider the homeomorphism $f : \mathbf{H} \to \mathbf{H}$ given by $f(z) = z + \mathrm{Re}(z)$. Prove that the hyperbolic length of a path in $\mathbf{H}$ (measured with respect to the element of arc-length $\frac{1}{\mathrm{Im}(z)}\,|\mathrm{dz}|$) is not invariant under f.

Does there exist any non-zero real valued function g so that the hyperbolic length of a path in $\mathbf{H}$ (measured with respect to the element of arc-length $\frac{1}{\mathrm{Im}(z)}\,|\mathrm{dz}|$) is not invariant under $f(z) = z + g(\mathrm{Re}(z))$?

Answer
Consider $f + o : [0,1] \longrightarrow \mathbf{H},\ f_0(t) = t + (1+t)i \quad f_0'(t) = 1 + i$

$$\mathrm{length}_{\mathbf{H}}(f_0) = \int_0^1 \frac{\sqrt{2}}{1+t}\,dt = \sqrt{2}\ln(2).$$

$$f \circ f_0(t) = f_0(t) + \mathrm{Re(f_0(t))} = \mathrm{2t + (1+t)i} \quad \mathrm{(f \circ f_0)'(t) = 2 + i}.$$

$$\mathrm{length}_{\mathbf{H}}(f \circ f_0) = \int_0^1 \frac{\sqrt{5}}{1+t}\,dt = \sqrt{5}\ln(t) \neq \mathrm{length}_{\mathbf{H}}(f_0).$$

Suppose now that there exists $g : \mathbf{R} \longrightarrow \mathbf{R}$ so that, for all paths $f_0 : [a,b] \longrightarrow \mathbf{H}$ so that

$$\mathrm{length}_{\mathbf{H}}(f_0) = \mathrm{length}_{\mathbf{H}}(f \circ f_0)$$

Write

$$\begin{aligned} f_0(t) &= x(t) + iy(t) \\ f \circ f_0(t) &= x(t) + g(x(t)) + iy(t) \end{aligned}$$

$$\begin{aligned} \mathrm{length}_{\mathbf{H}}(f_0) &= \int_a^b \frac{1}{y(t)}\sqrt{(x'(t))^2 + (y'(t))^2}\,dt \\ \mathrm{length}_{\mathbf{H}}(f \circ f_0) &= \int_a^b \frac{1}{y(t)}\sqrt{(x'(t))^2(1 + s'(x(t)))^2 + (y'(t))^2}\,dt \end{aligned}$$

consider f_0 with $y(t) = 1$ so that $y'(t) = 0$, hence

$$\int_a^b \sqrt{(x'(t))^2}\,dt = \int_a^b \sqrt{(x'(t))^2(1 + g'(x(t)))^2}\,dt$$

using an argument similar to the one given in class, since this is true for all intervals $[a,b] \subseteq \mathbf{R}$ and all g, we have that $g'(x) = 0$ all x and so g is constant (and g constant is a Möbius transformation).

[Actually, have $(1 + g'(x(t)))^2 = 1$ so either

$1 + g'(x(t)) = 1$ in which case $g'(x(t)) = 0$ and so g is constant, or

$1 + g'(x(t)) = -1$ so $g'(x(t)) = -2$ and so $g(x) = 2x + c$

So, either g is constant (Möbius transformation) or $g(x) = -2x + c$ in which case $z + g(\mathrm{Re}(z)) = -\bar{z} + c$ again in Möb($\mathbf{H}$).]