Question

For integers $m \geq 0$, consider the paths $g_m : [0,1] \to \mathbf{H}$ given by

$$g_m(t) = t + (t^{3m} + 1)i.$$

Write down the integral giving the hyperbolic length of the curve $g_m([0,1])$ in $\mathbf{H}$. Evaluate it if you can.

By considering what the curves $g_m([0,1])$ look like in $\mathbf{H}$ as $m \to \infty$, determine the putative limit of the hyperbolic lengths $\text{length}_{\mathbf{H}}(g_m([0,1]))$ as $m \to \infty$.

Answer

$\text{Im}(g_m) = t^{3m} + 1$

$|g_m'(t)| = |1 + 3mt^{3m-1}i| = \sqrt{1 + 9m^2 t^{6m-2}}$

So $\text{length}_{\mathbf{H}}(g_m) = \displaystyle\int_0^1 \frac{1}{t^{3m}+1}\sqrt{(1 + 9m^2 t^{6m-2}}\, dt.$

$\underline{m=0}$ $\text{length}_{\mathbf{H}}(g_0) = \displaystyle\int_0^1 \frac{1}{1+1}\sqrt{1+0}\, dt = \frac{1}{2}$

$\underline{m=1}$ $\text{length}_{\mathbf{H}}(g_1) = \displaystyle\int_0^1 \frac{1}{t^3+1}\sqrt{1+9t^4}\, dt$

and others that I don't know how to evaluate.

But, as $m \to \infty$, $g + m([0,1])$ approaches the union of the horizontal Euclidean line segment from i to $i+1$ and the vertical line segment from $1+i$ to $1+2i$.

So, this horizontal Euclidean line segment is parametrized by $f : [0,1] \longrightarrow \mathbf{H}$, $f(t) = t + i$ and so $\text{length}_{\mathbf{H}}(f) = \displaystyle\int_0^1 dt = 1$

and the vertical line segment has $\text{length}_{\mathbf{H}} = \ln(2) = d_{\mathbf{H}}(1+i, 1+2i)$.

So $\underline{\text{length}_{\mathbf{H}}(g_m) \longrightarrow 1 + \ln(2)}$ as $m \to \infty$.