Question

Set $G = \mathrm{stab}_{\text{Möb}}(\infty) = \{m \in \text{Möb} \mid m(\infty) = \infty\}$. Write down a set of generating elements for G.

Let $\eta(z)\,|\mathrm{dz}|$ be an element of arc-length on $\mathbf{C}$ which is invariant under G. Prove that η is constant.

Answer

If $m(z) = \dfrac{az+b}{cz+d}$ $\in$Möb$^+$ and $m(\infty) = \infty$, then $c = 0$, and so $m(z) = \alpha z + \beta$.
Generators of this are $\{P_\beta(z) = z + \beta | \beta \in \mathbf{C}$ and $\mathrm{L}_\alpha(\mathrm{z}) = \alpha\mathrm{z} | \alpha \in \mathbf{C} - \{0\}\}$.
Note that $C(z) = \bar{z}$ also fixes ∞, and so if $n(z) = \dfrac{a\bar{z}+b}{c\bar{z}+d}$ fixes ∞, then $C \circ n(z)$ fixes ∞, and so is as above.
So, generators for G are:

$P_\beta(z) = z + \beta;\ \beta \in \mathbf{C}$

$L_\alpha(z) = \alpha(z);\ \alpha \in \mathbf{C} - \{0\}$

$C(z) = \bar{z}$

$\underline{\eta(z)|\,dz|}$ invariant under G, $\ f : [a, b] \longrightarrow \mathbf{C}$ a path

$$\begin{aligned}
\text{length(f)} &= \int_f \eta(z)|\,dz| \\
&= \int_{P_\beta \circ f} \eta(z)|\,dz| \\
&= \int_a^b \eta(P_\beta \circ f(t))|(P_\beta \circ f)'(t)|\,dt \\
&= \int_a^b \eta(f(t) + \beta)|f'(t)|\,dt \\
&= \int_a^b \eta(P_\beta \circ f(t))|f'(t)|\,dt \\
&= \int_f \eta \circ P_\beta(z)|dz|.
\end{aligned}$$

and so $\eta(z) = \eta \circ P_\beta(z) = \eta(z + \beta)$ all $\beta \in \mathbf{C}$. So, given $\omega,\ z \in \mathbf{C}$,set $\beta = \omega - z$ and note that $\eta(z) = \eta(z + \beta) = \eta(\omega)$ and to η is constant.