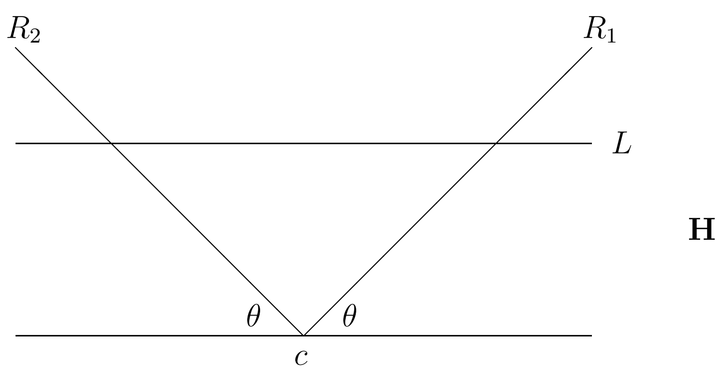
Question

Given any angle θ in the interval $(0, \frac{\pi}{2})$ and any point c in $\mathbf{R}$, consider the two Euclidean rays R_1 and R_2 in $\mathbf{H}$, originating at c and making angles θ and $\pi - \theta$ with the positive real axis, respectively. Show that the hyperbolic distance between the two points $R_1 \cap L$ and $R_2 \cap L$ is independent of L, where L is any horizontal Euclidean line in $\mathbf{H}$.

Answer



First involve symmetry; since the picture is invariant under reflection through the vertical line through c, which is a consequence of the choice of angles of $R_1 R_2$ with $\mathbf{R}$, not only are the imaginary points of the two points $R_1 \cap L$ and $R_2 \cap L$ equal, but both lie on a circle centred at c. So, the hyperbolic line segment from $p_1 = R_1 \cap L$ to $p_2 = R_2 \cap L$ is parametrized by

$$f : [\theta, \pi - \theta] \longrightarrow \mathbf{H}, \quad f(t) = c + pe^{it} \quad p = |c - p_1| = |c - p_2|$$

Then, $\mathrm{Im}(f(t)) = p \sin(t)$ (since $c \in \mathbf{R}$) and $|f'(t0| = p$, and so $\mathrm{length}_{\mathbf{H}}(f) = d_{\mathbf{H}}(p_1 p_2) = \int_{\theta}^{\pi-\theta} \frac{1}{\sin(t)}\, dt$, which is independent of L, and in fact depends only on the angles.