Question

A viscous fluid has constant density ρ and constant kinematic viscosity ν. Non-dimensionalise the steady Navier-Stokes equations with suitable scalings to show that, when the Reynolds number Re satisfies

$$Re \ll 1,$$

the non-dimensional equations of motion become, to leading order, the "slow-flow" equations

$$\begin{aligned} \nabla p &= \nabla^2 \underline{q} \\ \nabla . \underline{q} &= 0 \end{aligned}$$

where p and $\underline{q}$ denote the fluid pressure and velocity respectively. Give an example of a fluid mechanics scenario wher such a "slow-flow" approximation would apply.

The fluid is contained between two plates. Cartesian coordinates (x, z) are used. The plates are situated at $z = 0$ and $z = \delta h(x)$ in dimensionless variables where $h(x)$ may be regarded as a given function and $\delta \ll 1$. Show how, by further scaling the slow flow equations according to

$$\begin{aligned} z &= \delta Z \\ w &= \delta W \\ p &= \frac{P}{\delta^2}, \end{aligned}$$

the two-dimensional "lubrication theory" equations

$$\begin{aligned} P_x &= u_{ZZ} \\ P_Z &= 0 \\ u_x + W_Z &= 0 \end{aligned}$$

may be derived. Given the conditojs under which this limit is valid.

The top plate is held fixed, and the bottom plate is moved with a non-dimensional speed of 1 in the positive x-direction. Show that the pressure satisfies Reynolds' equation

$$\frac{d}{dx}\left(\frac{h^3}{6}\frac{dP}{dx}\right) - \frac{dh}{dx} = 0$$

and give suitable boundary conditions for this differential equation. Suppose now that $h(x) = a + bx$ where a and b are constants. WITHOUT SOLVING the equation comment briefly on whether b should be positive or negative if the top plate is to support a load.

Answer

We have $\left.\begin{array}{c}(\underline{q}.\nabla)\underline{q} = -\dfrac{1}{\rho}\nabla p + \nu\nabla^2\underline{q} \\ div(\underline{q}) = 0\end{array}\right\}$

Since slow flow, we set $\underline{x} = L\overline{\underline{x}}$, $p = (\mu U/L)\overline{p}$, $\underline{q} = U\overline{\underline{q}}$, $\Rightarrow$

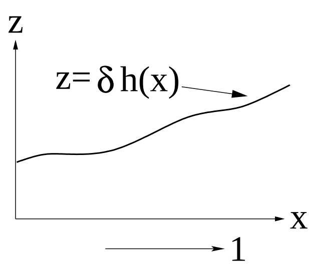
$$\left.\begin{array}{c}\dfrac{U^2}{L}(\overline{\underline{q}}.\nabla)\overline{\underline{q}} = -\dfrac{1}{\rho L}\overline{\nabla}\overline{p}\left(\dfrac{\mu U}{L}\right) + \dfrac{\nu U}{L^2}\overline{\nabla}^2\overline{\underline{q}} \\ \overline{\nabla}.\overline{\underline{q}} = 0\end{array}\right\}$$

Multiplying the momentum equation by $\nu U/L^2$

$\Rightarrow \dfrac{LU}{\nu}(\overline{\underline{q}}.\overline{\nabla})\overline{\underline{q}} = -\overline{\nabla}\overline{p} + \overline{\nabla}^2\overline{\underline{q}}$, but $\dfrac{LU}{\nu} = Re$

Thus $Re(\overline{\underline{q}}.\nabla)\overline{\underline{q}} = -\overline{\nabla}\overline{p} + \overline{\nabla}^2\overline{\underline{q}}$, $\Rightarrow$ for $Re \ll 1$ we get, to lowest order, $\overline{\nabla}\overline{p} = \overline{\nabla}^2\overline{\underline{q}}$, $\overline{\nabla}.\overline{\underline{q}} = 0$

Any decent flow example will do, eg sperm swimming, treacle flowing, tar running down a telegraph pole.



We have (drop bars)

$$\left.\begin{array}{c}p_x = u_{xx} + u_{zz} \\ p_z = w_{xx} + w_{zz} \\ u_x + w_z = 0\end{array}\right\}$$

So further scale $\dfrac{1}{\delta^2}P$, $w = \delta W$, $z = \delta Z$.

$$\Rightarrow \left.\begin{array}{c}\dfrac{1}{\delta^2}P_x = u_{xx} + \dfrac{1}{\delta^2}U_{ZZ} \\ \dfrac{1}{\delta}P_Z = \delta W_{xx} + \dfrac{1}{\delta}W_{ZZ} \\ u_x + W_Z = 0\end{array}\right\}$$

Now to leading order as $\delta \to 0$, we must get

$$P_x = u_{ZZ}, \quad P_Z = 0, \quad u_x + W_Z = 0$$

Need $\delta \ll 1$, $\delta^2 Re \ll 1$.

Now the top plate is fixed and the bottom is moved with speed 1.

we have $u_{ZZ} = P_x \Rightarrow u = \dfrac{Z^2 P_x}{2} + AZ + B$

Now $u = 0$ on $Z = h$, $U = 1$ on $Z = 0$

$\Rightarrow B = 1, \quad o = \dfrac{h^2 P_x}{2} + Ah + 1$

and so

$$\begin{aligned} u &= 1 + \frac{Z^2 P_x}{2} + Z\left(-\frac{1}{h} - \frac{hP_x}{2}\right) \\ &= \frac{Z(Z-h)P_x}{2} + \left(1 - \frac{Z}{h}\right) \end{aligned}$$

$\int_h^0 (u_x + W_Z)\, dZ = 0 \Rightarrow \int_0^h u_x\, dZ = 0$ (since $W = 0$ on $z = 0, h$)

$\Rightarrow \frac{\partial}{\partial x}\int_0^h u\, dZ = 0$ (since $u = 0$ on $Z = h$)

$\frac{d}{dx}\left[\frac{Z^3}{6}P_x - \frac{Z^2 h}{4}P_x + Z - \frac{Z^2}{2h}\right]_0^h \Rightarrow \frac{d}{dx}\left[-\frac{h^3}{12}P_x + \frac{h}{2}\right] = 0$

$\Rightarrow \frac{d}{dx}\left(\frac{h^3}{6}P_x\right) - \frac{dh}{dx} = 0$

We impose $P = 0$ at $x = 0, 1$

When $h = a + bx$ need $b < 0$ for $P > 0$ so that fluid is being forced into a NARROWING gap.