Question

Incompressible viscous fluid of constant density ρ and constant kinematic viscosity ν occupies the region

$$\{-\infty < x < \infty, -\infty < y < \infty\}$$

above an impermeable plane situated at $z = 0$. This plane creates flow in the region $z > 0$ by moving with a speed $U\cos(\omega t)$ parallel to itself, where U and ω are both constant. There are no body forces and no pressure gradient.

(i) What flow would be produced if this experimetn was carried out with an *inviscid* fluid?

(ii) By seeking time dependent solutions to the Navier-Stokes equations of the form

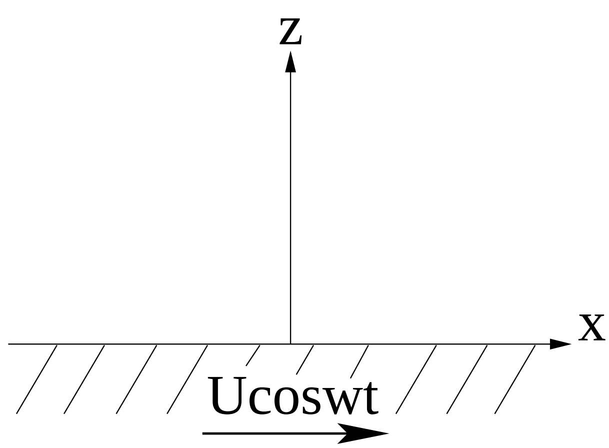
$$\underline{q} = \begin{pmatrix} RE|f(z)\exp(i\alpha t)| \\ 0 \\ 0 \end{pmatrix}$$

where f is a suitably chosen function, α is a suitably chosen constant and RE signifies that the real part of the expression is to be taken, show that

$$u = U\exp\left(-z\sqrt{\frac{\omega}{2\nu}}\right)\cos\left(\omega t - z\sqrt{\frac{\omega}{2\nu}}\right).$$

(iii) Sketch the velocity profile for this flow at time $t = 0$.

Answer



(Assume wlog that the plane moves parallel to the x-axis)

(i) For inviscid fluid there would be NO MOTION AT ALL since an inviscid fluid can sustain slip at the walls.

(ii) Seek a solution $\underline{q} = (u(z,t), 0, 0)$. Then $div(\underline{q}) = 0$ so that mass conservation is satisfied.

The momentum equations give

$$\begin{aligned} u_t = uu_x + vu_y + wu_z &= 0 + \nu(u_{xx} + u_{yy} + u_{zz}) \\ 0 + uv_x + vv_y + wv_z &= 0 + \nu(0 + 0 + 0) \\ 0 + uw_x + vw_y + ww_z &= 0 + \nu(0 + 0 + 0) \end{aligned}$$

i.e. The second two give just $0 = 0$ and the first reduces to

$u_t = \nu u_{zz}$ with $u(0,t) = U \cos \omega t$.

Now trying $u = f(z)e^{i\alpha t}$ (and taking real parts throughout) we see first from the B/C that $\alpha = \omega$, $f(0) = U$. Also

$f(z)i\alpha e^{i\alpha t} = \nu f'' e^{i\alpha t} \Rightarrow f'' - \dfrac{i\alpha}{\nu} f = 0$

$\Rightarrow f = Ae^{\lambda_1 z} + Be^{\lambda_2 z} \quad (\lambda^2 = i\alpha/\nu = i\omega/\nu)$

Thus $\lambda = \sqrt{\dfrac{\omega}{\nu}}\sqrt{i}$. Now $i = e^{i\frac{\pi}{2} + 2k\pi} \Rightarrow \sqrt{i} = e^{i\frac{\pi}{4} + k\pi}$

i.e. the two roots of i are

$$\begin{aligned} e^{i\frac{\pi}{4}} &= \cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4} &&= \tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}} \\ and \quad e^{5i\frac{\pi}{4}} &= \cos\tfrac{5\pi}{4} + i\sin\tfrac{5\pi}{4} &&= \tfrac{-1}{\sqrt{2}} - \tfrac{i}{\sqrt{2}} \end{aligned}$$

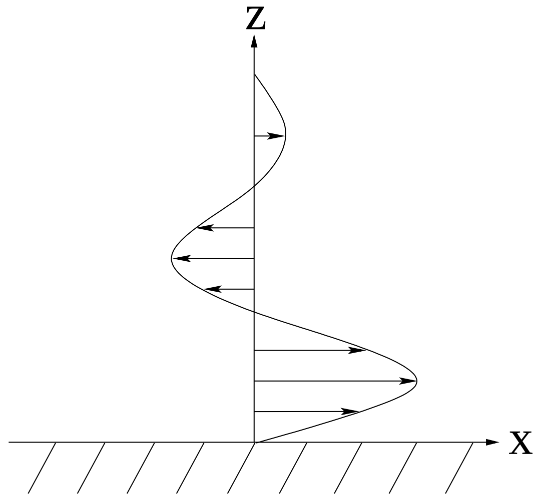
So for finiteness as $z \to +\infty$ need $\lambda_2 = (\dfrac{-1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}})\sqrt{\dfrac{\omega}{\nu}}$

$\Rightarrow f = A\exp((-1-i)z\sqrt{\tfrac{\omega}{2\nu}})$ now $f(0) = 0 \Rightarrow A = U$

$\Rightarrow$

$$\begin{aligned} u &= U \exp\left[(-1-i)z\sqrt{\frac{\omega}{2\nu}} + i\omega t\right] \\ &= U \exp\left(-z\sqrt{\frac{\omega}{2\nu}}\right) \exp\left(i\left(\omega t - z\sqrt{\frac{\omega}{2\nu}}\right)\right) \end{aligned}$$

(iii)



(For full marks must show alternation, decay and periodicity)