Question

A viscous fluid of constant density ρ and constant dynamic viscosity μ flows steadily under gravity down a rigid impermeable plane. The acceleration due to gravity is denoted by g. The plane is inclined at an angle α to the horizontal. The flow is two-dimensional and the coordinate origin is in the plane. The x-axis is taken to be parallel to the plane (along the line of greatest slope) and the y-axis is normal to the plane. The fluid velocity is denoted by $\underline{q} = (u, v)$. Show that, at any point in the fluid, the shear stress τ (i.e. the stress in the x-direction exerted on a plane $y =$constant) is given by

$$\tau = \mu(u_y + v_x)$$

On the top surface of the fluid the pressure is given by p_a and there is no shear stress. By assuming a flow velocity $\underline{q}$ of the form

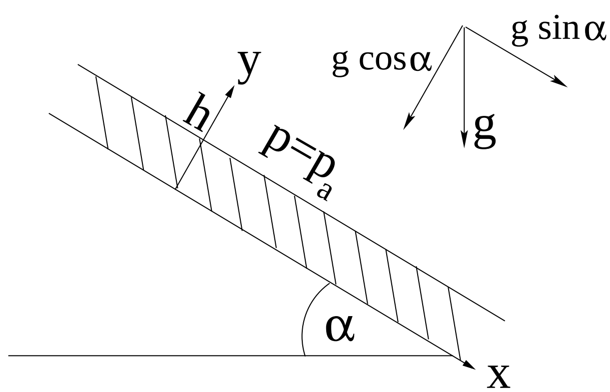
$$\underline{q} = (u(y), 0)^T,$$

determine a solution to the Navier-Stokes equations that represents unidirectional flow down the plane in a layer of constant thickness h.

Show that the momentum flux M of fluid flowing down the plane is given by

$$M = \frac{2h^5\rho^3 g^2 \sin^2\alpha}{12\mu^2}$$

Answer



The stress vector $\underline{t}$ is given by $T\hat{\underline{n}}$.

Now $T = -p\delta_{ij} + 2\mu d_{ij}$ i.e. $T = \begin{pmatrix} -p + 2\mu u_x & \mu(u_y + v_x) \\ \mu(u_y + v_x) & -p + 2\mu v_y \end{pmatrix}$

Now the normal to a plane $y = constant$ is $(0, 1)$, so

$\underline{t} = \begin{pmatrix} -p + 2\mu u_x & \mu(u_y + v_x) \\ \mu(u_y + v_x) & -p + 2\mu v_y \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mu(u_y + v_x) \\ -p + 2\mu v_y \end{pmatrix}$
And thus the shear (i.e. in the x-direction) stress is $\mu(u_y + v_x)$.
Now by Navier-Stokes

$$\begin{aligned} uu_x + vu_y &= -p_x/\rho + \nu(u_{xx} + u_{yy}) + g\sin\alpha \\ uv_x + vv_y &= -p_y/\rho + \nu(v_{xx} + v_{yy} - g\cos\alpha \\ u_x + v_y &= 0 \end{aligned}$$

Now assume that $\underline{q} = (u(y), 0)^T$. Then $u_x + v_y = 0$ and we get

$$\begin{aligned} 0 &= -p_x/\rho + \nu(u_{yy}) + g\sin\alpha \\ 0 &= -p_y/\rho + \nu(0) - g\cos\alpha \end{aligned}$$

so $p_y = -\rho g\cos\alpha$, $\Rightarrow p = -\rho gy\cos\alpha + K$. Now on $y = h$ we have $p = p_a$ $\Rightarrow p = p_a + \rho g(h - y)\cos\alpha$.
Thus $p_x = 0$ and so
$u_{yy} = \dfrac{-g\sin\alpha}{\nu} \Rightarrow u_y = \dfrac{-gy\sin\alpha}{\nu} + C$
Now since $\tau = \nu(u_y + v_x)$ and $v = 0$, we have (since the shear stress is 0 on $y = h$)
$u_y = 0$ on $y = h$
$\Rightarrow u_y = \dfrac{g(h - y)\sin\alpha}{\nu}$
$\Rightarrow u = \dfrac{g(hy - y^2/2)\sin\alpha}{\nu} + C'$
But by no slip $u = 0$ on $y = 0$
$\Rightarrow u = \dfrac{g(hy - y^2/2)\sin\alpha}{\nu}$, $p = p_a + \rho g(h - y)\cos\alpha$
Now momentum $= \rho u$ $\Rightarrow$ momentum flux $= \rho u^2$
$\Rightarrow M = \int_0^h \rho u^2\, dy = \int_0^h \dfrac{\rho g^2 \sin^2\alpha}{\nu^2} \left(h^2 y^2 - hy^4 + \dfrac{y^4}{4} \right) dy$
$\Rightarrow$

$$\begin{aligned} M &= \frac{\rho g^2 \sin^2\alpha}{\nu^2} \left[\frac{h^2 y^2}{3} - \frac{hy^4}{4} + \frac{y^5}{20} \right]_0^h \\ &= \frac{\rho^3 g^2 \sin^2\alpha}{\mu^2} [h^5](1/3 - 1/4 + 1/20) \end{aligned}$$

$\Rightarrow M = \dfrac{2\rho^3 g^2 \sin^2\alpha}{15\mu^2}$