Question

BRIEFLY explain the difference between the *Lagrangian* and *Eulerian* dscripitons of flow. A fluid occupies the region $D(t)$, and at a time $t =)$ position vectores in $D(0)$ are described by $\underline{X} = (X, Y, Z)$. At a later time t position vectors in the fluid are given by $\underline{x} = \underline{r}(\underline{X}, t)$ where

$$\begin{aligned} x &= X + Yt^2 \\ y &= Y + Yt^3 \\ z &= Z + 2tZ \end{aligned}$$

Determine bothe the Eulerian description of the flow (in the form $\underline{q} = \underline{q}(\underline{x}, t)$ where $\underline{q} = \partial \underline{x}/\partial t$) and the "inverse Lagrangian" description of the flow in the form $\underline{X} = \underline{r}^{-1}(\underline{x}, t)$. Is this an incompressible flow?

Prove that, if d/dt denotes time derivative with $\underline{X}$ fixed and $\partial/\partial t$ denotes a time derivatice with $\underline{x}$ fixed, the the "convective differentiation" formula

$$\frac{d\underline{g}}{dt} = \frac{\partial \underline{g}}{\partial t} + (\underline{q}.\nabla)\underline{g}$$

holds for any suitably differentiable vector function $\underline{g}$.

Verify the convective derivative formula for the motion considered in the first part of the question when

$$\underline{g} = \begin{pmatrix} xt \\ yt^2 \\ z \end{pmatrix}.$$

Answer

LAGRANGIAN Move with the flow, hold $\underline{X}$ constant. Seek to determine $\underline{X} = \underline{X}(\underline{x}, t)$.

EULERIAN Fix attention on one spot in space; seek to determine velocity $\underline{q} = \underline{q}(\underline{x}, t)$

Now $\begin{array}{l} x = X + Yt^2 \\ y = Y(1+t^3) \\ z = Z(1+2t) \end{array}$ $\quad \underline{q} = \dfrac{\partial \underline{x}}{\partial t} = \left.\dfrac{\partial \underline{x}}{\partial t}\right|_{\underline{X}} = \begin{pmatrix} 2Yt \\ 3t^2Y \\ 2Z \end{pmatrix}$

But we need $\underline{q}(\underline{x}, t)$ not $\underline{q}(\underline{X}, t)$.

Now also we have

$Y = \dfrac{y}{1+t^3}, \quad Z = \dfrac{z}{1+2t}, \quad \Rightarrow X = x - Yt^2 = x - \dfrac{yt^2}{1+t^3}$

Thence

$$\underline{q} = \begin{pmatrix} 2yt/(1+t^3) \\ 3yt^2/(1+t^3) \\ 2z/(1+2t) \end{pmatrix}$$

$$rmand \quad \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} x - yt^2/(1+t^3) \\ y/(1+t^3) \\ z/(1+2t) \end{pmatrix}$$

Now $div(\underline{q}) = 0 + \dfrac{3t^2}{1+t^3} + \dfrac{2}{1+2t} \neq 0$

Now if $\dfrac{d}{dt}$ means 'fix $\underline{X}$' we have, for any suitably differentiable $\underline{g}$,

$$\begin{aligned} \frac{\partial \underline{g}}{\partial t} &= \frac{\partial \underline{g}}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial \underline{g}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \underline{g}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \underline{g}}{\partial z}\frac{\partial z}{\partial t} \\ &= \underline{g}_t + \underline{g}_x u + \underline{g}_y v + \underline{g}_z w \end{aligned}$$

(where, as usual, $\underline{q} = (u, v, w) = \dfrac{\partial \underline{x}}{\partial t}$)

Thus $\dfrac{d\underline{g}}{dt} = \underline{g}_t + (\underline{q}.\nabla)\underline{q}$

Now we have $\underline{g}_t = \begin{pmatrix} x \\ 2yt \\ 0 \end{pmatrix}, \quad (\underline{q}.\nabla)\underline{g} = \begin{pmatrix} 2yt^2/(1+t^3) \\ 3yt^4/(1+t^3) \\ 2z/(1+2t) \end{pmatrix}$ and so

$$\underline{g}_t + (\underline{q}.\nabla)\underline{g} = \begin{pmatrix} x + 2yt^2/(1+t^3) \\ 2yt + 3yt^4/(1+t^3) \\ 2z/(1+2t) \end{pmatrix}$$

$$= \begin{pmatrix} x + 2yt^2/(1+t^3) \\ (2yt + 5yt^4)/(1+t^3) \\ 2z/((1+2t) \end{pmatrix}$$

Now

$$\underline{g} = \begin{pmatrix} Xt + Yt^3 \\ Y(1+t^3)t^2 \\ Z(1+2t) \end{pmatrix}$$

$$\Rightarrow \frac{d\underline{g}}{dt} = \begin{pmatrix} X + 3Yt^2 \\ Y(2t+5t^4) \\ 2Z \end{pmatrix}$$

$$= \begin{pmatrix} x - \frac{yt^2}{1+t^3} + \frac{3yt^2}{1+t^3} \\ \frac{2ty}{1+t^3} + \frac{5yt^4}{1+t^3} \\ \frac{2z}{1+2t} \end{pmatrix} = \begin{pmatrix} x + \frac{2yt^2}{1+t^3} \\ \frac{2ty + 5yt^4}{1+t^3} \\ \frac{2z}{1+2t} \end{pmatrix} = \underline{q}_t + (\underline{q}.\nabla)\underline{g}$$