Question

Find the general solution of the following differential equations

(i) $x\dfrac{dy}{dx} + y = 1$

(ii) $\dfrac{dy}{dx} = 2xy$

(iii) $y(x+3) + x(3-y)\dfrac{dy}{dx} = 0$

(iv) $\dfrac{dy}{dx} = \dfrac{1-y}{1+x}$

(v) $y\sin x\dfrac{dy}{dx} = \cos x + \sin x$

(vi) $3e^{x+y}\dfrac{dy}{dx} = 2$

Answer

(i)

$$x\frac{dy}{dx} + y = 1.$$

Rearrange to get $\dfrac{dy}{dx} = \dfrac{1-y}{x}$

Variables separable $\leftarrow$ $\Rightarrow \left(\dfrac{1}{1-y}\right)\dfrac{dy}{dx} = \dfrac{1}{x}$

Thus

$$\begin{aligned} & \int \frac{dy}{(1-y)} = \int \frac{dx}{x} \\ \Rightarrow\quad & -\ln(1-y) = \ln(x) + c, \qquad \text{letc} = \ln \text{k} \\ \Rightarrow\quad & \ln[(1-y)^{-1}] = \ln(kx) \\ \Rightarrow\quad & \frac{1}{1-y} = kx \\ \Rightarrow\quad & \underline{kx(1-y) = 1} \end{aligned}$$

(ii)

$\frac{dy}{dx} = 2xy$. Rearrange to get $\frac{1}{y}\frac{dy}{dx} = 2x$

$$\text{variables separable} \quad \begin{aligned} \Rightarrow \int \frac{dy}{y} &= \int 2x\,dx \\ \Rightarrow \ln y &= x^2 + c \\ \Rightarrow y &= e^{x^2+c} \end{aligned}$$

(iii)

$y(x+3) + x(3-y)\frac{dy}{dx} = 0$

Rearrange to get,

$$\left(\frac{3-y}{y}\right)\frac{dy}{d} + \frac{x+3}{x} = 0,$$

i.e., variables separable. Thus

$$\begin{aligned}
\int \frac{3-y}{y}\,dy &= -\int dx \frac{(x+3)}{x} \\
\text{or} \quad \int \left(1-\frac{3}{y}\right) dy &= \int \left(1+\frac{3}{x}\right) dx \\
\Rightarrow \quad y - 3\ln y &= x + 3\ln x + c \\
\text{setc} = 3\ln \text{k} & \\
\Rightarrow \quad y - x &= 3\ln y + 3\ln x + 3\ln k \\
\Rightarrow \quad (y-x) &= 3\ln xyk \\
\Rightarrow \quad (kxy)^3 &= e^{y-x}
\end{aligned}$$

(iv)

$\dfrac{dy}{dx} = \dfrac{1-y}{1+x}$.

Rearrange to get

$$\frac{1}{1-y}\frac{dy}{dx} = \frac{1}{(1+x)} \Rightarrow \text{variables separable}$$

$$\begin{aligned}
\Rightarrow \quad \int \frac{dy}{1-y} &= \int \frac{dy}{1+x} \\
\Rightarrow \quad -\ln(1-y) &= \ln(1+x) + c \\
\text{set c} = \ln \text{k} & \\
\Rightarrow \quad \ln\left[\frac{1}{(1-y)}\right] &= \ln(x+1) + \ln k \\
\Rightarrow \quad \frac{1}{(1-y} &= k(x+1) \\
\Rightarrow \quad k(x+1)(1-y) &= 1
\end{aligned}$$

(v)

$y \sin x \dfrac{dy}{dx} = \cos x + \sin x$

Rearrange to get

$$y\frac{dy}{dx} = \frac{\cos c + \sin x}{\sin x}$$

i.e., variables separable

Remember: $\cot x = \dfrac{\cos x}{\sin x} = \dfrac{1}{\tan x}$

$$\begin{aligned} \Rightarrow \int y\,dy &= \int dx(1+\cot x) \\ \frac{y^2}{2} &= x + \ln\sin x + c \end{aligned}$$

$\int \cot x\,dx = \ln\sin x$ standard integral

(vi)

$3e^{x+y}\dfrac{dy}{dx} = 2.$

Rearrange to get $3e^{y}\dfrac{dy}{dx} = 2e^{-x}$

variables separable.

Thus

$$\begin{aligned} 3\int e^{y}\,dy &= 2\int e^{-x}\,dx \\ \Rightarrow \quad 3e^{y} &= -2e^{-x} + c \\ \Rightarrow \quad 3e^{x+y} + 2 &= ce^{x} \end{aligned}$$