Question

Find the general solution of the following differential equations

(i) $xy\dfrac{dy}{dx} = y^2 - x^2$

(ii) $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + y^2}$

(iii) $x(y+x)\dfrac{dy}{dx} = y(y-x)$

(iv) $x^2\dfrac{dy}{dx} = xy - y^2$

Answer

(i)

$xy\dfrac{dy}{dx} = y^2 - x^2$ is of the type $P(x,y)\dfrac{dy}{dx} + Q(x,y) = 0$ with

$$\left.\begin{array}{lll} P(x,y) = xy & \Rightarrow & \text{degree2} \\ Q(x,y) = x^2 - y^2 & \Rightarrow & \text{degree2} \end{array}\right\} \Rightarrow \underline{\text{homogenous}}$$

Thus use $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ (where $v = v(x)$)

and divide both sides through by x^2 etc......

$$\begin{aligned} xy\frac{dy}{dx} &= y^2 - x^2 \\ \Rightarrow \frac{dy}{dx} &= \frac{y^2 - x^2}{xy} \\ & \quad \text{divide top and bottom of RHS by x}^2 \\ \Rightarrow \frac{dy}{dx} &= \frac{(\frac{y}{x})^2 - 1}{(\frac{y}{x})} \end{aligned}$$

Substitute $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ on LHS and $y = vx$ on RHS

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{v^2 - 1}{v} \\ \Rightarrow \quad x\frac{dv}{dx} &= \frac{v^2 - 1}{v} - v \\ \text{or} \quad x\frac{dv}{dx} &= -\frac{1}{v} \end{aligned}$$

This is now variables separable. Hence:

$$
\begin{aligned}
\int v\,dv &= -\int \frac{dx}{x} \\
\Rightarrow \frac{v^2}{2} &= -\ln x + c \\
&= -\ln x - \ln k \\
\Rightarrow \frac{v^2}{2} &= -\ln kx \\
&= \ln\left(\frac{1}{kx}\right) \\
\Rightarrow v^2 &= 2\ln\left(\frac{1}{kx}\right) \\
\text{or v}^2 &= \ln\left(\frac{1}{k^2x^2}\right)
\end{aligned}
$$

Now put back $y = vx$

$$
\begin{aligned}
\frac{y^2}{x^2} &= \ln\left(\frac{1}{k^2x^2}\right) \\
\text{or} \quad \frac{1}{k^2x^2} &= e^{(\frac{y}{x})^2} \\
\text{or} \quad x^2 &= \frac{1}{k^2}e^{-(\frac{y}{x})^2}
\end{aligned}
$$

(ii) This is of the form $P\dfrac{dy}{dx} + Q = 0$

where $\left.\begin{array}{lcl} P(x,y) = x^2 + y^2 & \Rightarrow & \text{degree2} \\ Q(x,y) = -xy & \Rightarrow & \text{degree2} \end{array}\right\} \Rightarrow \underline{\text{homogenous}}$

Thus set $y = vx$, $\Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Divide top and bottom by x^2:

$$\frac{dy}{dx} = \frac{(\frac{xy}{x^2})}{(\frac{x^2+y^2}{x^2})} = \frac{(\dfrac{y}{x})}{[1 + (\frac{y}{x})^2]}$$

or

$$\begin{array}{rrcl} & v + x\dfrac{dv}{dx} & = & \dfrac{v}{1+v^3} \\ \Rightarrow & x\dfrac{dv}{dx} & = & \dfrac{v}{1+v^3} - v \\ & & = & \dfrac{v - v - v^3}{1+v^2} \\ & & = & -\dfrac{v^3}{1+v^2} \end{array}$$

This is now variables separable. Thus

$$\begin{array}{rrcl} & \displaystyle\int dv\frac{(1+v^2)}{-v^3} & = & \displaystyle\int \frac{dx}{x} \\ \Rightarrow & \displaystyle -\int \frac{dv}{v^3} - \int \frac{dv}{v} & = & \displaystyle\int \frac{dx}{x} \\ \Rightarrow & \dfrac{1}{2v^2} - \ln v & = & \ln x + c \\ & c = \ln k & & \\ \Rightarrow & \dfrac{1}{2v^2} & = & \ln v + \ln x + \ln k \\ & & = & \ln(kvx) \end{array}$$

Thus $kvx = e^{\dfrac{1}{2v^2}}$.

Now replace $y = vx$ to get $\underline{ky = e^{\frac{x^2}{2y^2}}}$.

(iii) This is of the form $P(x,y)\dfrac{dy}{dx} + Q(x,y) = 0$

where $\left.\begin{array}{lll} P(x,y) = xy + x^2 & \Rightarrow & \text{degree2} \\ Q(x,y) = y^2 - yx & \Rightarrow & \text{degree2} \end{array}\right\} \Rightarrow \underline{\text{homogenous}}$

Thus rearrange to get

$$\frac{dy}{dx} = \frac{y^2 - xy}{xy + x^2}$$

Divide RHS top and bottom by x^2

$$\frac{dy}{dx} = \frac{(\frac{y^2}{x^2} - \frac{xy}{x^2})}{(\frac{xy}{x^2} - \frac{x^2}{x^2})} = \frac{(\frac{y}{x})^2 - (\frac{y}{x})}{(\frac{y}{x}) + 1}$$

Use $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\begin{aligned} \Rightarrow \quad v + x\frac{dv}{dx} &= \frac{v^2 - v}{v+1} \\ \Rightarrow \quad x\frac{dv}{dx} &= \frac{v^2 - v}{v+1} - v \\ &= \frac{v^2 - v - v^2 - v}{v+1} = \frac{-2v}{v+1} \end{aligned}$$

Thus

$$x\frac{dv}{dx} = -\frac{2v}{v+1}$$

This is variables separable, so

$$\begin{aligned} \int dv \left(\frac{v+1}{v}\right) &= -2\int \frac{dx}{x} \\ \Rightarrow \quad \int dv + \int \frac{dv}{v} &= -2\int \frac{dx}{x} \\ \Rightarrow \quad v + \ln v &= -2\ln x + c \\ \Rightarrow \quad \frac{y}{x} + \ln(\frac{y}{x}) &= -2\ln x + c \\ \Rightarrow \quad \frac{y}{x} &= -\ln\left(\frac{y}{x}\right) - 2\ln x + c \\ &= -\ln y + \ln x - 2\ln x + c \\ &= -\ln y - \ln x + c \\ & \quad c = -\ln k \\ \frac{y}{x} &= -\ln(yxk) \end{aligned}$$

$\Rightarrow \underline{y + x\ln(kxy) = 0}$

(iv) $x^2\dfrac{dy}{dx} = xy - y^2$

This is of the form $P(x,y)\dfrac{dy}{dx} + Q(x,y) = 0$

where $\left.\begin{array}{lcl} P(x,y) = x^2 & \Rightarrow & \text{degree2} \\ Q(x,y) = y^2 - xy & \Rightarrow & \text{degree2} \end{array}\right\} \Rightarrow \underline{\text{homogenous}}$

Thus rearrange to get

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}$$

Divide RHS top and bottom by x^2

$$\frac{dy}{dx} = \frac{\left(\frac{xy}{x^2} - \frac{y^2}{x^2}\right)}{\left(\frac{x^2}{x^2}\right)} = \frac{\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2}{1}$$

Set $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Thus

$$\begin{array}{rrcl} & v + x\dfrac{dv}{dx} & = & v - v^2 \\ \text{or} & x\dfrac{dv}{dx} & = & -v^2 \end{array}$$

This is variables separable

$$\begin{array}{lrcl} & \displaystyle\int \frac{dv}{v^2} & = & \displaystyle -\int \frac{dx}{x} \\ \Rightarrow & -\dfrac{1}{v} & = & -\ln x + c \\ \text{or} & \dfrac{1}{v} & = & \ln x - c \\ & & & \text{set c} = -\ln \text{k} \\ \Rightarrow & \dfrac{y}{x} & = & \ln x - c \\ \Rightarrow & \dfrac{y}{x} & = & -\ln kx \end{array}$$

or $\underline{y = x\ln kx}$