Question

The rate at which a liquid runs from a container is proportional to the square root of the depth h of the opening below the surface of the liquid. A cylindrical petrol storage tank is sunk in the ground with its axis vertical. There is a leak in the tank at an unknown depth. The level of the petrol in the tank, originally full, is found to drop by 20 cm in 1 hour, and by 19 cm in the next hour. Show that the differential equation modelling the leak is of the form

$$-\frac{dh}{dt} = kh^{\frac{1}{2}}, \quad k = \text{constant}$$

Solve this with the given data to find the depth at which the leak is located.

Answer

PICTURE

Let height of petrol level above hole be h. Rate of flowing out $= -\frac{dh}{dt}$ (negative because h is decreasing with t)

$\Rightarrow \propto \sqrt{h}$

Proportional to "$\propto$" is the same as "$= k \times \cdots$" where k is unknown at this stage

$$\text{Thus} \quad \frac{dh}{dt} = -k\sqrt{h}$$

$$\text{or} \quad \frac{dh}{dt} = -kh^{\frac{1}{2}} \quad \text{variables separable;}$$

$$\int \frac{dh}{h^{\frac{1}{2}}} = -\int k\,dt$$

$$\Rightarrow \quad 2h^{\frac{1}{2}} = -kt + c \ (\star)$$

Now when $h = 0$ the petrol is at the level of the hole. Thus when $h = 0$,

$$t = \frac{c}{k}$$

Thus we must find c, k. To do this use the two bits of information given. Let the height of the petrol above the hole at $t = 0$ be H.

Thus ($\star$) gives

$$2H^{\frac{1}{2}} = c$$

(1)
Now when $t = 1$ hr, $h = (H - 20)$cm

$$\Rightarrow 2(H - 20)^{\frac{1}{2}} = -k + c$$

(2)
and when $t = 2$hrs, $h = H - 20 - 19 = (H - 39)$cm

$$\Rightarrow 2(H - 39)^{\frac{1}{2}} = -2k + c$$

(3)
We must find H. If we know c, we know H from (1). Thus consider $2 \times (2) - (3)$:

$$4(H - 20)^{\frac{1}{2}} = -2k + 2c$$

$$\underline{2(H - 39)^{\frac{1}{2}} = -2k + c}$$

$$4(H - 20)^{\frac{1}{2}} - 2(H - 39)^{\frac{1}{2}} = c$$

But from (1) we have $c = 2H^{\frac{1}{2}}$, thus

$$4(H - 20)^{\frac{1}{2}} - 2(H - 39)^{\frac{1}{2}} = 2H^{\frac{1}{2}}$$

This looks difficult to solve, but isn't really.

$$2(H - 20)^{\frac{1}{2}} - (H - 39)^{\frac{1}{2}} = H^{\frac{1}{2}}$$

Square both sides:
$\left[2(H - 20)^{\frac{1}{2}} - (H - 39)^{\frac{1}{2}}\right]^2 = 4(H-20) + (H-39) - 4(H-20)^{\frac{1}{2}}(H-39)^{\frac{1}{2}} = H$
Thus
$4H - 80 + H - 39 - H = 4(H - 20)^{\frac{1}{2}}(H - 39)^{\frac{1}{2}}$
$\dfrac{4H - 119}{4} = (H - 20)^{\frac{1}{2}}(H - 39)^{\frac{1}{2}}$

Square again:

$$\left(\frac{4H-119}{4}\right)^2 = (H-20)(H-39)$$

$$\Rightarrow H^2 - \frac{2\times 119}{4}H + \left(\frac{119}{4}\right)^2 = H^2 - 20H - 39H + (20\times 39)$$

$$\Rightarrow -\frac{119}{2}H + \left(\frac{119}{4}\right)^2 = -59H + 780$$

$$\Rightarrow \left(59 - \frac{119}{2}\right)H = 780 - \left(\frac{119}{4}\right)^2$$

$$\Rightarrow -\frac{H}{2} = 780 - \left(|ds\frac{119}{4}\right)^2$$

Thus $H = \left[\left(\frac{119}{4}\right)^2 - 780\right]\times 2cm = 210.125cm$

$$\underline{H \approx 2.1m}$$