Question

When a thermometer is placed in a liquid, the rate at which the indicated temperature θ rises is proportional to the difference between the indicated temperature and the true temperature T of the liquid. Initially the thermometer indicates $15°C$; 30 seconds later the indicated temperature is $20°C$, and a further 30 seconds later it is $21°C$. Show that the differential equation governing the rise of the thermometer reading is

$$\frac{d\theta}{dt} = k(\theta - T), \quad k = \text{constant}$$

Solve this with the given data to determine the true temperature of the liquid, which is assumed to be constant.

Answer

Let true temperature be T. Let thermometer reading be θ, time be t.

$$\text{At} \quad \begin{array}{ll} t = 0 & \theta = 15°C \\ t = 30secs & \theta = 20°C \\ t = 60secs & \theta = 21°C \end{array}$$

What is the differential equation?

$$\begin{array}{rrcl} & \text{rateoftemprise} & \propto & \theta - T \\ \Rightarrow & \frac{d\theta}{dt} & \propto & \theta - T \\ \Rightarrow & \frac{d\theta}{dt} & = & k(\theta - T) \end{array}$$

k=constant of proportionality

This is variables separable

Thus

$\int \frac{d\theta}{\theta - T} = k \int dt \Rightarrow \ln(\theta - T) = kt + c$

Must find k, c and then T. Apply boundary conditions.

At

$$\begin{array}{l} t = 0 \Rightarrow \ln(15 - T) = c \quad (1) \\ t = 30secs \Rightarrow \ln(20 - T) = 30k + c \quad (2) \\ t = 60secs \Rightarrow \ln(21 - T) = 60k + c \quad (3) \end{array}$$

As before, consider $2 \times (2) - (3)$

$$2\ln(20 - T) = 60k + 2c$$

$$\underline{\ln(21 - T) = 60k + c}$$

$$2\ln(20 - T) - \ln(21 - T) = c$$

Thus with (1) we have

$$2\ln(20-T)-\ln(21-T)=\ln(15-T)$$

$$\text{or }\ \ln[(20-\mathrm{T})^2]-\ln(21-\mathrm{T})-\ln(15-\mathrm{T})=0$$

or $\ln\left[\dfrac{(20-T)^2}{(21-T)(15-T)}\right]=0$
$\Rightarrow \dfrac{(20-T)^2}{(21-T)(15-T)}=1$
$\Rightarrow (20-T)^2=(21-T)(15-T)$
$\Rightarrow 400+T^2-40T=315-36T$
$\Rightarrow 85=4T$

$$\underline{\Rightarrow T=21.25^{\circ}C}$$