Question

A body falling *from rest* at $t = 0$, in a medium where the resistance is proportional to the velocity obeys the equation

$$\frac{dv}{dt} = g - kv$$

where k is constant. Show that eventually the body achieves a constant (terminal) velocity. If this terminal velocity is V, find, in terms of k, the time that elapses before a body falling from rest attains a velocity $\frac{V}{2}$.

Answer

$\frac{dv}{dt} = g - kv$

Variables separable

$\int \frac{dv}{g-kv} = \int dt \Rightarrow -\frac{1}{k}\ln(g-kv) = t + c$

What is c? Particle falls from rest $(v = 0)$ at $t = 0$.

Thus, $-\frac{1}{k}\ln(g-0) = 0 + c \Rightarrow -\frac{1}{k}\ln g = c$

Thus

$$\begin{aligned} -\frac{1}{k}\ln(g-kv) &= t - \frac{1}{k}\ln g \\ \text{or} \quad \frac{1}{k}\ln g - \frac{1}{k}\ln(g-kv) &= t \\ \Rightarrow \quad \frac{1}{k}\ln\left(\frac{g}{g-kv}\right) &= t \\ \text{or} \quad \frac{g}{g-kv} &= e^{kt} \\ ge^{-kt} &= g - kv \end{aligned}$$

$\Rightarrow \underline{v = \frac{g}{k}(1-e^{-kt})}$

Now as $t \to \infty$, $v \to \frac{g}{k}(1 - e^{-\infty})$

But $e^{-\infty} = 0 \Rightarrow \underline{v \to \frac{g}{k}}$ the terminal velocity

Thus $V = \frac{g}{k}$.

Now when $v = \frac{V}{2}$ we have

$$\begin{aligned}
& & \frac{V}{2} &= \frac{g}{k}(-e^{-kt}) \\
& or & \frac{g}{2k} &= \frac{g}{k}(1 - e^{-kt}) \\
& \Rightarrow & \frac{1}{2} &= 1 - e^{-kt} \\
& \Rightarrow & e^{-kt} &= \frac{1}{2}
\end{aligned}$$

$$\Rightarrow -kt = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow kt = \ln 2$$

$$\Rightarrow \underline{t = \frac{1}{k}\ln 2}$$