QUESTION

Describe briefly the transition diagram method of setting up the equations relating the steady-state probabilities of the different possible states of a queueing system.

Mass produced cars are 'customised' in a two step process. Process 1 involves mechanical alterations, followed by Process 2 where the trim is modified. Only one car at a time can be accepted by each Process; no queueing is allow between the two Processes. Thus a car finishing Process 1 must remain in it, blocking it, if there is still a car undergoing Process 2. A car finishing Process 1 proceeds immediately to Process 2, if this is free. Processing times are exponentially distributed with mean $1/\mu$ for both Processes. Cars arrive at rate λ, but if Process 1 is not free, cars do not wait but are turned away. Set up equations relating the equilibrium probabilities of the different states of the system, and hence find these probabilities. Find also

(i) L, the mean number of cars in the system,

(ii) F, the proportion of cars turned away.

Show that if $\rho = \lambda/\mu$ is small, then $L = 2\rho + O(\rho^2)$ and $F = \rho + O(\rho^2)$.

If the cost of turning each car away is C_0, whilst the cost per unit time of maintaining processing rate μ (for both Processes) is $C_1\mu$, find an approximate expression for ρ^*, the processing rate that minimises costs, assuming this is small.

ANSWER

Description of transition diagram method should mention:

(i) Use of nodes to represent different states of system

(ii) Links to represent possible transitions between states

(iii) An equation for each node equating the long term balance of transitions into and out of that node.

Car customizing problem:

Let (m,n)=(process 1 in state m, Process 2 in state n), $m = 0,1;\ n = 0,1$.

Transition diagram:

DIAGRAM

Let p_{mn}=prob(system in state (m,n)). Then

$$\left.\begin{array}{ll} (0,0): & \lambda p_{00} = \mu p_{01} \\ (0,1): & (\mu+\lambda)p_{01} = \mu p_{10} \\ (1,0): & \mu p_{10} = \lambda p_{00} + \mu p_{11} \end{array}\right\} \Rightarrow \begin{array}{l} p_{01} = \rho p_{00} \\ p_{10} = (1+\rho)p_{01} = \rho(1+\rho)p_{00} \\ p_{11} = p_{10} - \rho p_{00} = \rho^2 p_{00} \end{array}$$

Normalising condition gives:

$$p_{00} - [1 + \rho + \rho(1 + \rho) + \rho^2]^{-1} = [1 + 2\rho + 2\rho^2]^{-1}$$

$$\begin{aligned} L &= p_{01} + p_{10} + 2p_{11} \\ &= (2\rho + \rho^2 + 2\rho^2)(1 + 2\rho + 2\rho^2)^{-1} \\ &= 2\rho(1 + \frac{3}{2}\rho)(1 + 2\rho + 2\rho^2)^{-1} \\ &\approx 2\rho(1 + \frac{3}{2}\rho)(1 - 2\rho) + O(\rho^3) \\ &\approx 2\rho(1 - \frac{\rho}{2}) + O(\rho^3) \end{aligned}$$

$$F = p_{10} + p_{11} = (\rho + 2\rho^2)(1 - 2\rho) + O(\rho^3) \approx \rho + O(\rho^3$$

The average unit time cost is $K = \frac{C_0\lambda}{\mu} + C_1\mu$, assuming ρ remains small.

$$\frac{dK}{d\mu} \equiv -\frac{C_0\lambda}{\mu^2} + C_1 = 0 \text{ at } \mu = \sqrt{\frac{C_0\lambda}{C_1}}$$

$$\frac{d^2K}{d\mu^2} = 2\frac{C_0\lambda}{\mu^3} > 0 \text{ at } \mu = \sqrt{\frac{C_0\lambda}{C_1}}$$

giving a minimum.