Question

State the maximum Modulus Principle for a non-constant function $f(z)$ that is analytic within a simple closed contour γ and continuous on γ.
Show that if $f(z) \neq 0$ for all z within and on γ then the minimum value of $|f(z)|$ cannot be achieved in the interior of γ.
By considering the function $e^{f(z)}$ show that no non-constant harmonic function can achieve its maximum or minimum values in the interior of γ.
Hence find the maximum and minimum values of $x^3 - 3xy^2$ in the set $\{(x, y)|x^2 + y^2 \leq 1\}$ and find where the bounds are attained.

Answer

Statement of max mod. Proof of min mod applies max mod to $\frac{1}{f(z)}$ - bookwork

Let u be harmonic. Find a harmonic conjugate v so that $f(z) = u + iv$ is analytic.
Now $|e^{f(z)}| = e^u$ so $e^{f(z)} \neq 0$
Thus $|e^{f(z)}|$ achieves its max and min on γ.
Now e^u is an increasing function of u, so u achieves its max and min on γ.
Let $u(x, y) = x^3 - 3xy^2$, then $u_{xx} = 6x$ and $u_{yy} = -6x$, so u is harmonic. So the unit disc u achieves max and min on γ, i.e. where $y^2 = 1 - x^2$.
So $u(x, 1 - x^2) = x^3 - 3x(1 - x^2) = 4x^3 - 3x = h(x) \qquad -1 \leq x \leq 1$
$h'(x) = 12x^2 - 3 = 0$ where $4x^2 = 1$ i.e. $x = \pm\frac{1}{2}$

$$h''(x) = 24x \begin{array}{ll} > 0 \text{ at } x = \frac{1}{2} & - \text{ a local minimum} \\ < 0 \text{ at } x = -\frac{1}{2} & - \text{ a local maximum} \end{array}$$

$h(\frac{1}{2}) = -1 \qquad h(-\frac{1}{2}) = 1$
At the end points $h(-1) = -1 \qquad h(1) = 1$
so u achieves max values at $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ and $(1, 0)$
and u achieves min values at $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ and $(-1, 0)$.