Question

The function $f(z)$ is analytic inside and on a simple closed curve γ. Use Cauchy's theorem to prove Cauchy's integral formula that for b inside γ,

$$f(b) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-b}.$$

State a similar formula for the nth derivative $f^{(n)}(b)$.
Use these formulas to show that

i) $\displaystyle\int_\gamma \frac{e^2 z}{z^2}dz = 4\pi i$

ii) $\displaystyle\int_\gamma \frac{\sin z dz}{9z^2+1} = \frac{2\pi i}{3}\sinh\frac{1}{3}.$

where γ is the circle centre 0 and radius 1.

Answer

Proof of Cauchy's Integral formula - bookwork.

i) Let $f(z) = e^{2z} \quad f'(z) = 2e^{2z}$

$$\int_\gamma \frac{f(z)}{(z-0)^2}dz = 2\pi i f'(0) = 4\pi i$$

ii) $\displaystyle\frac{1}{z^2+\frac{1}{9}} = \frac{1}{\left(z+\frac{1}{3}i\right)\left(z-\frac{1}{3}\right)} = \frac{3}{2i}\left(\frac{1}{z-\frac{1}{3}i} - \frac{1}{z+\frac{1}{3}i}\right)$

$$\int_\gamma \frac{\sin z}{9z^2+1}dz = 2\pi i\frac{3}{2i}\frac{1}{9}\left\{\int_\gamma \frac{\sin z}{z-\frac{1}{3}i}dz - \int_\gamma \frac{\sin z}{z+\frac{1}{3}i}dz\right\}$$

$$= \frac{\pi}{3}\left(\sin\frac{1}{3}i - \sin\left(-\frac{1}{3}i\right)\right) = \frac{2\pi i}{3}\sinh\frac{1}{3}$$