Question
Write down the Euler-Langrange equation appropriate to the functional

$$I = \iint_Z F(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, x, y) dxdy$$

Find the Euler-Lagrange equations associated with the functionals

(i) $I = \iint (\bigtriangledown^2 u)^2 dxdy$

(ii) $I = \iint (u_{xx}u_{yy} - u_{xy}^2) dxdy$

Answer
The E-L equation is:
$0 = \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)$
$+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right) - \underbrace{\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial u_{xy}}\right)} + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial U_{yy}}\right)$
Assuming that both F and its first derivatives are given on the boundary. So for

(i) $F = (u_{xx} + u_{yy})^2 = (\bigtriangledown^2 u)^2 = F(u_{xx}, u_{yy})$ only, so E-L is:

$$\frac{\partial^2}{\partial x^2}[2(u_{xx} + u_{yy})] + \frac{\partial^2}{\partial y^2}[2(u_{xx} + u_{yy})] = 0$$

$\Rightarrow u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$
or $\underline{\bigtriangledown}^2(\underline{\bigtriangledown}^2 u) = 0$
or $\underline{\underline{\bigtriangledown^4 u = 0}}$

(ii) $F = (u_{xx}u_{yy} - u_{xy}^2) = F(u_{xx}, u_{yy}, u_{xy})$ only so E-L is:

$$\frac{\partial^2}{\partial x^2}(u_{yy}) + \frac{\partial^2}{\partial x \partial y}(-2u_{xy}) + \frac{\partial^2}{\partial y^2}(u_{xx}) = 0$$

$\Rightarrow un0 = 0!$

So F is such that I is <u>always</u> stationary!!!