Question

Write down the Euler-Lagrange equation $u(x,y)$ must satisfy on an area S of the $x,\ y$-plane if $u(x,y)$ takes prescribed values on the closed curve C bounding S and

$$I = \int_S dS(\bigtriangledown u)^r = \iint_S \{u_x^2 + u_y^2\}^{\frac{r}{2}} dxdy$$

is to be stationary, where r is a given real constant ($\neq 0$). (It may be assumed that $\bigtriangledown u \neq 0$ on S.

Answer

If $I = \iint (u_x^2 + u_y^2)^{\frac{r}{2}} dxdy$ the E-L equation is

$\dfrac{\partial F}{\partial u} - \dfrac{\partial}{\partial x}\left(\dfrac{\partial F}{\partial u_x}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\partial F}{\partial u_y}\right) = 0$ with $F = (u_x^2 + u_y^2)^{\frac{r}{2}}$

$\Rightarrow \dfrac{\partial}{\partial x}(ru_x(u_x^2 + u_y^2)^{\frac{r}{2}-1}) + \dfrac{\partial}{\partial y}(ru_y(u_x^2 + u_y^2)^{\frac{r}{2}-1}) = 0$

which, after a bit of tedious algebra can be written

$$[(r-1)u_x^2 + u_y^2]u_{xx} + 2(r-2)u_xu_yu_{xy} + [(r-1)u_y^2 + u_x^2]u_{yy} = 0$$