

Question

Find the partial differential equation satisfied by $u(x, y)$ if

$$I = \iint_S \{au_x^2 + 2bu_xu_y + cu_y^2 + eu^2\}dxdy$$

where a, b, c, e are functions of x , is to be stationary subject to $u(x, y)$ taking specified values on the boundary of S .

Answer

With $F = au_x^2 + 2bu_xu_y + cu_y^2 + eu^2$ the E-L equation is

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) = 0 \text{ with } a = a(x), b = b(x) \text{ etc.}$$

$$\Rightarrow \frac{\partial}{\partial x} (2au_x + 2bu_y) + \frac{\partial}{\partial y} (2bu_x + 2cu_y) - 2eu = 0$$

$$\Rightarrow au_{xx} + 2bu_{xy} + cu_{yy} + \frac{da}{dx}u_x + \frac{db}{dx}u_y - eu = 0$$