Exam Question
Topic: Reduction Formula
Let $I_n(x) = \int_1^x t(\ln t)^n \, dt$, where $x > 0$ and n is a non-negative integer.
(a) Show that $I_n(x) = \frac{x^2}{2}(\ln x)^n - \frac{n}{2} I_{n-1}(x)$, when $n \geq 1$.
(b) Find $I_0(x)$, $I_1(x)$ and $I_2(x)$.
(c) Find $\int_1^{\sqrt{\mathrm{e}}} t(\ln t)^4 \, dt$.

Solution

$$\begin{aligned}
\text{(a)} \quad I_n(x) = \int_1^x t(\ln t)^n \, dt &= \left[\frac{t^2}{2}(\ln t)^n\right]_1^x - \int_1^x \frac{t}{2} n(\ln t)^{n-1} \, dt \\
&= \frac{x^2}{2}(\ln x)^n - \frac{n}{2} I_{n-1}(x)
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad I_0(x) &= \frac{x^2}{2} - \frac{1}{2} \\
I_1(x) &= \frac{x^2}{2}(\ln x) + \left(\frac{-1}{2}\right)\left(\frac{x^2}{2}\right) + \left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right) = \frac{x^2}{2}(\ln x) - \frac{x^2}{4} + \frac{1}{4} \\
I_2(x) &= \frac{x^2}{2}(\ln x)^2 - \frac{2}{2} I_0(x) = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}(\ln x) + \frac{x^2}{4} - \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
\text{(c)} \quad I_n(\sqrt{\mathrm{e}}) &= \frac{\mathrm{e}}{2^n} - \frac{n}{2} I_{n-1}(\sqrt{\mathrm{e}}) \quad \textit{so} \\
I_4(\sqrt{\mathrm{e}}) &= \frac{\mathrm{e}}{32} - 2I_3(\sqrt{\mathrm{e}}) = \frac{\mathrm{e}}{32} - 2\left(\frac{\mathrm{e}}{16} - \frac{3}{2} I_2(\mathrm{e})\right) = \frac{\mathrm{e}}{32} - \frac{\mathrm{e}}{8} + 3I_2(\mathrm{e}) \\
&= \frac{\mathrm{e}}{32} - \frac{\mathrm{e}}{8} + 3\left(\frac{\mathrm{e}}{8} - \frac{\mathrm{e}}{4} + \frac{\mathrm{e}}{4} - \frac{1}{4}\right) = \frac{9\mathrm{e}}{32} - \frac{3}{4}.
\end{aligned}$$