Question

A model of a linear molecule consists of four masses m, $2m$, $2m$, m connected by springs with spring constant k. Denoting the displacement of the masses from their equlibrium positions by x_1, x_2, x_3, x_4 respectivily, the equations can be written in the matrix form as:

$$\frac{d}{dt^2}\mathbf{x} = A\mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \; A = \begin{pmatrix} -2b & 2b & 0 & 0 \\ b & -2b & b & 0 \\ 0 & 0 & 2b & -2b \end{pmatrix}, \text{ and } b = \frac{k}{2m}$$

The matrix A has eigenvalues $0, -b, -3b$ and $-4b$. Find all the corresponding eigenvectors of A, and hence show that the equations have three types of oscillatory solution.

(i) oscillations of frequency $\frac{1}{2\pi}\sqrt{\frac{k}{2m}}$ when $x_2 = \frac{1}{2}x_1$, $x_3 = -\frac{1}{2}x_1$, $x_4 = -x_1$,

(ii) oscillations of frequency $\frac{1}{2\pi}\sqrt{\frac{3k}{2m}}$ when $x_2 = x_3 = -\frac{1}{2}x_1$, $x_4 = x_1$,

(iii) oscillations of frequency $\frac{1}{\pi}\sqrt{\frac{k}{2m}}$ when $x_2 = -x_1$, $x_3 = x_1$, $x_4 = -x_1$,

Answer

$$\begin{pmatrix} \frac{d^2x_1}{dt^2} \\ \frac{d^2x_2}{dt^2} \\ \frac{d^2x_3}{dt^2} \\ \frac{d^2x_4}{dt^2} \end{pmatrix} = \begin{pmatrix} -2b & 2b & 0 & 0 \\ b & -2b & b & 0 \\ 0 & b & -2b & b \\ 0 & 0 & 2b & -2b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Eigenvalues are: 0, -b, -3b, -4b.
To find eigenvectors:

$$\underline{\lambda = 0} \qquad \text{Solve} \begin{pmatrix} -2b & 2b & 0 & 0 \\ b & -2b & b & 0 \\ 0 & b & -2b & b \\ 0 & 0 & 2b & -2b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{or} \quad \left.\begin{array}{rcl} -2bx_1 + 2bx_2 &=& 0 \\ bx_1 - 2bx_2 + bx_3 &=& 0 \\ bx_2 - 2bx_3 + bx_4 &=& 0 \\ 2bx_3 - 2bx_4 &=& 0 \end{array}\right\} \begin{array}{lr} \text{let} & x_1 = \alpha \\ \text{so} & x_2 = \alpha \\ \text{then} & x_3 = 2x_2 - x_1 = \alpha \\ \text{and} & x_4 = \alpha \end{array}$$

Suitable eigenvector $\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\lambda = -b}$ Solve $\begin{pmatrix} -b & 2b & 0 & 0 \\ b & -b & b & 0 \\ 0 & b & -b & b \\ 0 & 0 & 2b & -b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

or $\left.\begin{array}{rcl} -bx_1 + 2bx_2 &=& 0 \\ bx_1 - bx_2 + bx_3 &=& 0 \\ bx_2 - bx_3 + bx_4 &=& 0 \\ 2bx_3 - bx_4 &=& 0 \end{array}\right\}$ let $x_2 = \beta$, so $x_1 = 2\beta$, then $x_3 = x_2 - x_1 = -\beta$ and $x_4 = -2\beta$

Suitable eigenvector $\beta \begin{pmatrix} 2 \\ 1 \\ -1 \\ -2 \end{pmatrix}$

$\underline{\lambda = -3b}$ Solve $\begin{pmatrix} b & 2b & 0 & 0 \\ b & b & b & 0 \\ 0 & b & b & b \\ 0 & 0 & 2b & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

or $\left.\begin{array}{rcl} bx_1 + 2bx_2 &=& 0 \\ bx_1 + bx_2 + bx_3 &=& 0 \\ bx_2 + bx_3 + bx_4 &=& 0 \\ 2bx_3 - bx_4 &=& 0 \end{array}\right\}$ let $x_2 = \gamma$, so $x_1 = -2\gamma$, then $x_3 = -x_1 - x_2 = \gamma$ and $x_4 = -2\gamma$

Suitable eigenvector $\gamma \begin{pmatrix} -2 \\ 1 \\ 1 \\ -2 \end{pmatrix}$

$\underline{\lambda = -4b}$ Solve $\begin{pmatrix} 2b & 2b & 0 & 0 \\ b & 2b & b & 0 \\ 0 & b & 2b & b \\ 0 & 0 & 2b & 2b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

or $\left.\begin{array}{rcl} 2bx_1 + 2bx_2 &=& 0 \\ bx_1 + 2bx_2 + bx_3 &=& 0 \\ bx_2 + 2bx_3 + bx_4 &=& 0 \\ 2bx_3 + 2bx_4 &=& 0 \end{array}\right\}$ let $x_1 = \delta$, so $x_2 = -\delta$, then $x_3 = -x_1 - 2x_2 = \delta$ and $x_4 = -\delta$

Suitable eigenvector $\delta \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

The solutions are therefore:

1. Eigenvalue $\lambda = 0$ and eigenvector $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ gives a solution

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (B + Ct)$$

   (This solution is non - oscillatory)

2. Eigenvalue $\lambda = -b$ and eigenvector $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ -2 \end{pmatrix}$ gives a solution

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ -2 \end{pmatrix} (D \cos \sqrt{b}t + E \sin \sqrt{b}t)$$

   which oscillates with frequency $\dfrac{\sqrt{b}}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{2m}}$.
   (Recall: A solution of the form ($\mathbf{x} = D \cos \sqrt{b}t + E \sin \sqrt{b}t$ oscillates with time period $T = \frac{2\pi}{\omega}$ and frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$)

3. Eigenvalue $\lambda = -3b$ and eigenvector $\mathbf{x} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ -2 \end{pmatrix}$ gives a solution

$$\mathbf{x} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ -2 \end{pmatrix} (F \cos \sqrt{3b}t + G \sin \sqrt{3b}t)$$

   which oscillates with frequency $\dfrac{\sqrt{3b}}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{3k}{2m}}$

4. Eigenvalue $\lambda = -4b$ and eigenvector $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ gives a solution

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} (H \cos 2\sqrt{b}t + J \sin 2\sqrt{b}t)$$

which oscillates with frequency $\dfrac{2\sqrt{b}}{2\pi} = \dfrac{1}{\pi}\sqrt{\dfrac{k}{2m}}$.