Question

Write the second order differential equation

$$\frac{d^2x}{dt^2} = 3x + \frac{dx}{dt}$$

as a linear system of two first order equations, and solve it using matrix methods.

Answer

Put $y = \dfrac{dx}{dt}$ so that $\dfrac{dy}{dt} = \dfrac{d^2x}{dx^2} = 2x + \dfrac{dx}{dt} = 2x + y$ so we have a set of simultaneous first order differential equations

$\left.\begin{array}{rcl} \frac{dx}{dt} & = & y \\ \frac{dy}{dt} & = & 2x + y \end{array}\right\}$ or in matrix form $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$ so $\lambda = -1, 2$

$\underline{\lambda = -1}$ Solve $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $\left.\begin{array}{rcl} x + y & = & 0 \\ 2x + 2y & = & 0 \end{array}\right\}$ $\begin{array}{lrcl} \text{let} & x & = & \alpha \\ \text{so} & y & = & -\alpha \end{array}$

Suitable eigenvector $\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$

$\underline{\lambda = 2}$ Solve $\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $\left.\begin{array}{rcl} -2x + y & = & 0 \\ 2x - y & = & 0 \end{array}\right\}$ $\begin{array}{lrcl} \text{let} & x & = & \beta \\ \text{so} & y & = & 2\beta \end{array}$

Suitable eigenvector $\begin{pmatrix} \beta \\ 2\beta \end{pmatrix}$

General solution is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \beta \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$

Hence the general solution to the second order differential equation is $x = \alpha e^{-t} + \beta e^{2t}$