Question

Using your matrix calculations from question 3 above, write down the general solution to the following system of second order differential equations:

$$\frac{d^2x}{dt^2} = 2x + 3y$$

$$\frac{d^2y}{dt^2} = 2x + y.$$

Answer

In matrix form $\begin{pmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Eigenvalue $\lambda_1 = -1$ with eigenvector $\mathbf{x_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ contributes a solution $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} (A\cos t + B\sin t)$

Eigenvalue $\lambda_2 = 4$ with eigenvector $\mathbf{x_2} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ contributes a solution $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} (Ce^{2t} + De^{-2})$

The general solution to the equations is formed by adding these two solutions:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} (A\cos t + B\sin t) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} (Ce^{2t} + De^{-2}) \\ &= \begin{pmatrix} A\cos t + B\sin t + 3Ce^{2t} + 3De^{-2t} \\ -A\cos t - B\sin t + 2Ce^{2t} + 2De^{-2t} \end{pmatrix} \end{aligned}$$