Question

Find the eigenvalues and normalised eigenvectors for each of the following matrices. In each case, write down an orthogonal matrix R such that R^TAR is a diagonal matrix (you should verify this by calculating R^TAR):

$$\text{(i) } A = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}; \text{ (ii) } B = \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}; \text{ (iii) } C = \begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Answer

(i)

$$\begin{aligned} \begin{vmatrix} 2-\lambda & -4 \\ -4 & 8-\lambda \end{vmatrix} &= (2-\lambda)(8-\lambda) - 16 \\ &= \lambda^2 - 10\lambda \\ &= \lambda(\lambda - 10) = 0 \\ \text{so} \quad \lambda &= 0, 10 \end{aligned}$$

$\underline{\lambda = 0}$ Solve $\begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\text{or} \quad \left.\begin{aligned} 2x - 4y &= 0 \\ -4x + 8y &= 0 \end{aligned}\right\} \begin{aligned} &\text{let} & y &= \alpha \\ &\text{so} & x &= 2\alpha \end{aligned}$$

Suitable eigenvector $\begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix}$ which normalises to $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

$\underline{\lambda = 10}$ Solve $\begin{pmatrix} -8 & -4 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\text{or} \quad \left.\begin{aligned} -8x - 4y &= 0 \\ -4x - 2y &= 0 \end{aligned}\right\} \begin{aligned} &\text{let} & x &= \beta \\ &\text{so} & y &= -2\beta \end{aligned}$$

Suitable eigenvector $\begin{pmatrix} \beta \\ -2\beta \end{pmatrix}$ which normalises to $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{pmatrix}$

Take the orthogonal matrix $R = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix}$ with $R^T = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix}$

[Note: check that the eigenvectors are orthogonal using the dot product: $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{pmatrix} = \frac{2}{5} - \frac{2}{5} = 0$]

Then

$$\begin{aligned} \mathrm{AR} &= \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{10}{\sqrt{5}} \\ 0 & \frac{-20}{\sqrt{5}} \end{pmatrix} \\ \mathrm{R^TAR} &= \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 0 & \frac{10}{\sqrt{5}} \\ 0 & \frac{-20}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix} \end{aligned}$$

as required

(ii)

$$\begin{aligned} \begin{vmatrix} 4-\lambda & 5 \\ 5 & 4-\lambda \end{vmatrix} &= (4-\lambda)^2 - 25 \\ &= \lambda^2 - 8\lambda - 9 \\ &= (\lambda-9)(\lambda+1) = 0 \\ \text{so} \quad \lambda &= -1, 9 \end{aligned}$$

$\underline{\lambda = -1}$ Solve $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $5x + 5y = 0$ $\quad$ let $x = \alpha$, so $y = -\alpha$

Suitable eigenvector $\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$ which normalises to $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$

$\underline{\lambda = 9}$ Solve $\begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $\left.\begin{aligned} -5x + 5y &= 0 \\ 5x - 5y &= 0 \end{aligned}\right\}$ let $x = \beta$, so $y = \beta$

Suitable eigenvector $\begin{pmatrix} \beta \\ \beta \end{pmatrix}$ which normalises to $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Take the orthogonal matrix $R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ with $R^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

Then

$$\begin{aligned} \mathrm{AR} &= \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{9}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{9}{\sqrt{2}} \end{pmatrix} \\ \mathrm{R^TAR} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{9}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{9}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 9 \end{pmatrix} \end{aligned}$$

as required

(iii)

$$\begin{vmatrix} 5-\lambda & 3 & 0 \\ 3 & 5-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} 5-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & 0 \\ 0 & 4-\lambda \end{vmatrix} + 0$$

$$\begin{aligned} &= (5-\lambda)^2(4-\lambda) - 9(4-\lambda) \\ &= (\lambda-4)(\lambda^2-10\lambda+16) \\ &= (\lambda-4)(\lambda-2)(\lambda-8) = 0 \\ \text{so} \quad \lambda &= 2, 4, 8 \end{aligned}$$

$\underline{\lambda = 2}$ Solve $\begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

or $\left.\begin{aligned} 3x + 3y &= 0 \\ 2z &= 0 \end{aligned}\right\}$ $\begin{aligned} z &= 0 \\ \text{let} \quad x &= \alpha \\ \text{so} \quad y &= -\alpha \end{aligned}$

Suitable eigenvector $\begin{pmatrix} \alpha \\ -\alpha \\ 0 \end{pmatrix}$ which normalises to $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix}$

$\underline{\lambda = 4}$ Solve $\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

or $\left.\begin{aligned} x + 3y &= 0 \quad (1) \\ 3x + y &= 0 \quad (2) \end{aligned}\right\}$ (2) - 3 (1) gives $-8y = 0 \Rightarrow y = 0 \Rightarrow x = 0$.
Let $z = \beta$

Suitable eigenvector $\begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}$ which normalises to $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda = 8}$ Solve $\begin{pmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

or $\left.\begin{aligned} -3x + 3y &= 0 \\ 3x - 3y &= 0 \\ 2z &= 0 \end{aligned}\right\}$ $\begin{aligned} z &= 0 \\ \text{let} \quad x &= \gamma \\ \text{so} \quad y &= \gamma \end{aligned}$

Suitable eigenvector $\begin{pmatrix} \gamma \\ \gamma \\ 0 \end{pmatrix}$ which normalises to $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$

Take the orthogonal matrix $R = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$

with $R^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

Then

$$\begin{aligned} \mathrm{AR} &= \begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{2}} & 0 & \frac{8}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} & 0 & \frac{8}{\sqrt{2}} \\ 0 & 4 & 0 \end{pmatrix} \\ \mathrm{R^T AR} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{2}} & 0 & \frac{8}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} & 0 & \frac{8}{\sqrt{2}} \\ 0 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} \end{aligned}$$

as required