QUESTION For each of the following quadratic forms, use eigenvalues and eigenvectors to rotate the axes in order to identify what type of conic it represents.

(a) $9x^2 - 4xy + 6y^2 - 10x - 20y - 5 = 0$,
(b) $3x^2 - 8xy - 12y^2 - 30x - 64y = 0$,
(c) $4x^2 - 20xy + 25y^2 - 15x - 6y = 0$,
(d) $9x^2 + 12xy + 4y^2 - 52 = 0$.

ANSWER In matrix form the equation is
$\mathbf{x}^t A\mathbf{x} - \begin{bmatrix} 10 & 20 \end{bmatrix} \mathbf{x} - 5 = 0$
where
$A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$
The eigenvalues of A are 5 (eigenvector col(1,2)) and 10 (eigenvector col(2,-1)). There are several different orthogonal matrices which can be used, half have det=1 so are pure rotation matrices (the others have det=-1 so reverse orientation as well). Any one will do to identify the type of conic. Putting $\mathbf{x} = P\underline{\xi}$ where
$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \qquad \underline{\xi} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$
the equation becomes
$10\xi^2 + 5\eta^2 - 10\sqrt{5}\eta - 5 = 0$, or $2\xi^2 + \eta^2 - 2\sqrt{5}\eta - 1 = 0$
It is also possible to get any of
$2\xi^2 + \eta^2 + 2\sqrt{5}\eta - 1 = 0 \quad 2\eta^2 + \xi^2 - 2\sqrt{5}\xi - 1 = 0$
$2\eta^2 + \xi^2 + 2\sqrt{5}\xi - 1 = 0$
This may also be written $\frac{\xi^2}{3} + \frac{(\eta-\sqrt{5})^2}{6} = 1$ which is an ellipse.
The equation $3x^2 - 8xy - 12y^2 - 30x - 64y = 0$ has eigenvalues 4 (eigenvector col(4,-1)) and -13 (eigenvector col(1,4)). The equation can be transformed to :
$4\xi^2 - 13\eta^2 - \frac{56}{\sqrt{17}}\xi - \frac{286}{\sqrt{17}}\eta = 0$
or to $4(\xi - \frac{7}{\sqrt{17}})^2 - 13(\eta + \frac{11}{\sqrt{17}})^2 = -81$
which is a hyperbola.
The equation $4x^2 - 20xy + 25y^2 - 15x - 6y = 0$ has eigenvalues) (eigenvector col(5,2)) and 29 (eigenvector col(-2,5)).The equation can be transformed to :
$29\xi^2 - \frac{87}{\sqrt{29}}\eta = 0$ and to $\eta^2 - \frac{3}{\sqrt{29}}\eta = 0$
which is a parabola.
The equation $9x^2 - 20xy + 4y^2 - 52 = 0$ has eigenvalues 0 (eigenvector col(2,-3)) and 29 (eigenvector col(3,2)). The equation can be transformed to :
$\eta^2 = 4$
which is a pair of parallel straight lines.