Question
Consider the functions

$$\begin{aligned} g(z) &= \mathrm{Log(z)}, \\ h(z) &= \mathrm{Log(-z)}, \\ f(z) &= g(z) - h(z), \end{aligned}$$

(i) Show that $g(z)$ is defined with $\arg(z)$ on $(-\pi, \pi]$ and that $h(z)$ is defined with $\arg(z)$ on $(-2\pi, 0]$. (Hint: note the difference between $rmArg$ and arg).

(ii) Draw the branch cut which makes the function $f(z)$ single valued and continuous.

(iii) Calculate $\dfrac{df}{dz}$. What does this suggest about the function $f(z)$?

(iv) Calculate $f(i)$.

(v) Calculate $f(-i)$.

(vi) Do your answers to (ii) and (iii) contradict your answers to (iii)? (Hint: Consider a general point with $Im(z) > 0$ and then one with $Im(z) < 0$).

Answer

(i) $g(z)$ $\mathrm{Log|z| + iArg(z)}$
so $\mathrm{Arg(z)} \in (-\pi, \pi]$ by definition

$$\begin{aligned} h(z) &= \log|-z| + i\mathrm{Arg(-z)} \\ &= \log|z| + i\mathrm{Arg(-z)} \end{aligned}$$

$$\text{so} \quad \begin{array}{rcccl} -\pi &<& \arg(z) &\leq& \pi \\ -\pi &<& \arg(ze^{i\pi}) &\leq& \pi \\ -\pi &<& \arg(z) + \pi &\leq& \pi \\ -2\pi &<& \arg(z) &\leq& 0 \end{array}$$

(ii) $f(z) = \log(z) - \mathrm{Log}(-\mathrm{z})$

The branch cut for $\mathrm{Log}(z)$ is

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The branch cut for $\mathrm{Log}(-z)$ is

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Putting them together:

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(iii)

$$\begin{aligned} \frac{df}{dz} &= \frac{d}{dz}[\mathrm{Log(z)}] - \frac{\mathrm{d}}{\mathrm{dz}}[\mathrm{Log(-z)}] \\ &= \frac{1}{z} - \frac{(-1)}{(-z)} = \frac{1}{z} - \frac{1}{z} = 0 \end{aligned}$$

This suggests that $\underline{f\text{=constant}}$ for all z.

(iv)

$$\begin{aligned} f(i) &= \mathrm{Log(i)} - \mathrm{Log(-i)} \\ &= \log|i| + i\mathrm{Arg(i)} - \log|\mathrm{i}| - \mathrm{iArg(-i)} \\ &= i(\mathrm{Arg(i)} - \mathrm{Arg(-i)}) \\ &= i\left(\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right) \\ &= \underline{+i\pi} \end{aligned}$$

(v)

$$\begin{aligned} f(-i) &= \mathrm{Log(-i) - Log(i)} \\ &= \log|-i| + i\mathrm{Arg(-i)} - \log|+\mathrm{i}| - \mathrm{iArg(i)} \\ &= i(\mathrm{Arg(-i) - Arg(i)}) \\ &= i\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) \\ &= \underline{-i\pi} \end{aligned}$$

(vi) At first sight (iv) $\neq$ (v) which contradicts (iii).

Resolution is: The cut in (ii) extends along all of the real axis , partitioning $Im(z) > 0$ from $Im(z) < 0$. The derivative is correst

but $\begin{cases} f = const(1) \text{ for Im(z)} > 0 \\ f = const(2) \text{ for Im(z)} < 0 \end{cases}$

This can be seen for arbitrary points:

Take $z = a + ib$ with $\underline{b > 0}$.

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$$\begin{aligned} f(a+ib) &= \log|a+ib| + i\mathrm{Arg(a+ib)} \\ &\quad - \log|-a-ib| - i\mathrm{Arg(-a-ib)} \\ &= i[\mathrm{Arg(a+ib) - Arg(-a-ib)}] \\ &= i[\theta - (-\pi + \theta)] \\ &= \underline{i\pi} \end{aligned}$$

Now take $z = x + iy$ with $\underline{y > 0}$.

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$$
\begin{aligned}
f(x+iy) &= \log|x+iy| + i\mathrm{Arg(x+iy)} \\
&\quad - \log|-x-iy| - i\mathrm{Arg(-x-iy)} \\
&= i[\mathrm{Arg(x+iy)} - \mathrm{Arg(-x-iy)}] \\
&= i[\alpha - (\pi - \alpha)] \\
&= \underline{-i\pi}
\end{aligned}
$$