Question
Find the general solution of the differential equation:

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = e^{-t} + t$$

Answer

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = e^{-t} + t$$

CF:

$$\begin{aligned} m^2 - m - 2 &= 0 \\ (m-2)(m+1) &= 0 \\ m &= 2,\ -1 \\ x_c &= Ae^{-t} + Be^{2t} \end{aligned}$$

PI: x_1^* is the solution of $\frac{d^2x_1^*}{dt^2} - \frac{dx_1^*}{dt} - 2x_1 = t$
Try $x_1^* = Pt + Q$, so $\frac{dx_1^*}{dt} = P$ and $\frac{d^2x_1^*}{dt^2} = 0$
Therefore

$$\frac{d^2x_1^*}{dt^2} - \frac{dx_1^*}{dt} - 2x_1^* = -P - 2(Pt + Q) \quad \equiv \quad t$$

$\Rightarrow -2Pt - (-2Q - P) \equiv t$
$\Rightarrow 2Q - P = 0 \Rightarrow 2P = 1 \Rightarrow P = -\frac{1}{2}$ and $Q = \frac{1}{4}$
Hence

$$x_1^* = \frac{1}{4}(-2t + 1)$$

x_2^* is the solution of $\frac{d^2x_2^*}{dt^2} - \frac{dx_2^*}{dt} - 2x = e^{-t}$
Can not try Re^{-t} because this is the same as the complementary function.
Try instead $x_2^* = Rte^{-t}$

$$\begin{aligned} x_2^* &= Rte^{-t} \\ \frac{dx_2^*}{dt} &= R(1-t)e^{-t} \\ \frac{d^2x_2^*}{dt^2} &= R(-1-(1-t))e^{-t} = R(t-2)e^{-t} \end{aligned}$$

Hence

$$\begin{aligned}
\frac{d^2x_2^*}{dt^2} - \frac{dx_2^*}{dt} - 2x_2^* &= Re^{-t}\{(t-2) - (t-1) - 2t\} \\
&= -3Re^{-t} \\
&\equiv e^{-t} \\
\Rightarrow R &= -\frac{1}{3}
\end{aligned}$$

Hence

$$x = \left(A - \frac{1}{3}t\right)e^{-t} + Be^{2t} + \frac{1}{4}(2t+1)$$