Question
Solve the initial value problem for the forced damped system

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = \cos 2t,$$

subject to $x(0) = 1$; $\dfrac{dx(0)}{dt} = 0$

Answer

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = \cos 2t$$

Auxiliary equation:

$$\begin{aligned} m^2 + 2m + 4 &= 0 \\ \Rightarrow (m+1)^2 + 3 &= 0 \\ \Rightarrow m &= -1 \pm \sqrt{3}j \end{aligned}$$

$$x_c(t) = e^{-t}[A\cos\sqrt{3}t + B\sin\sqrt{3}t]$$

PI:

$$\begin{aligned} x^*(t) &= P\cos 2t + Q\sin 2t \\ \frac{dx^*}{dt} &= 2Q\cos 2t - 2P\sin 2t \\ \frac{d^2x^*}{dt^2} &= -4P\cos 2t - 4Q\sin 2t \end{aligned}$$

$$\begin{aligned} \frac{d^2x^*}{dt^2} + 2\frac{dx^*}{dt} + 4x^* &= \cos 2t\{-4P + 4Q + 4P\} + \sin 2t\{-4Q - 4P + 4Q\} \\ &= 4Q\cos 2t - 4P\sin 2t \\ &\equiv \cos 2t \end{aligned}$$

Hence $P = 0\, Q = \dfrac{1}{4}$
General solution:

$$x(t) = e^{-t}[A\cos\sqrt{3}t + B\sin\sqrt{3}t] + \frac{1}{4}\sin 2t$$

$$\frac{dx}{dt} = e^{-t}[(\sqrt{3}B - A)\cos\sqrt{3}t - (B + \sqrt{3}A)\sin\sqrt{3}t] + \frac{1}{2}\cos 2t$$

$$x(0) = A = 1 \frac{dx(0)}{dt} = \sqrt{3}B - A + \frac{1}{2} = 0$$
$$\Rightarrow \sqrt{3}B - \frac{1}{2} = 0 \Rightarrow B = \frac{1}{2\sqrt{3}}$$

Hence

$$x(t) = e^{-t}\left[\cos\sqrt{3}t + \frac{1}{2\sqrt{3}}\sin\sqrt{3}t\right] + \frac{1}{4}\sin 2t$$