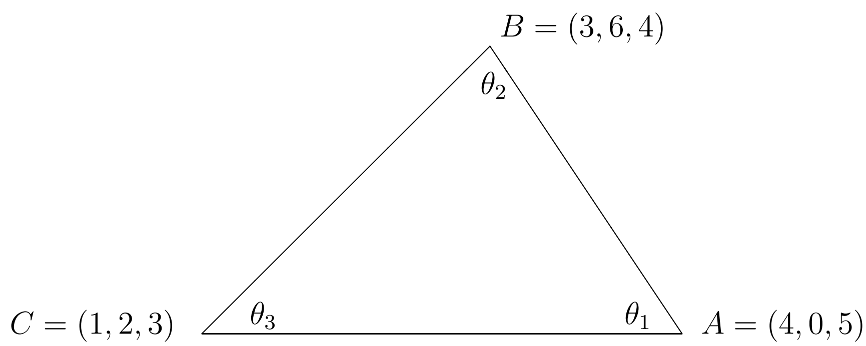
Question

Show that the triangle with vertices $(4,0,5)$, $(3,6,4)$ and $(1,2,3)$ has a right angle. (HINT: consider the edges of the triangle as vectors and calculate the angle between them.)

Answer

Let the triangle have vertices A, B and C:



The vertices A, B and C have position vectors

$\begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ respectively.

Find the vectors corresponding to the edges of the triangle:

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ -1 \end{pmatrix} \qquad \overrightarrow{BA} = \begin{pmatrix} 1 \\ -6 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix} \qquad \overrightarrow{CA} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -1 \end{pmatrix} \qquad \overrightarrow{CB} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

Remember: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if the dot product $\mathbf{u} \cdot \mathbf{v} = 0$.

Test θ_1: $\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{pmatrix} -1 \\ 6 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix} = 3 + 12 + 2 = 17 \neq 0$ so angle θ_1 is not a right angle.

Test θ_2: $\overrightarrow{BA} \cdot \overrightarrow{BC} = \begin{pmatrix} 1 \\ -6 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -4 \\ -1 \end{pmatrix} = -2 + 24 - 1 = 21 \neq 0$ so angle θ_2 is not a right angle.

Test θ_3: $\overrightarrow{CA} \cdot \overrightarrow{CB} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = 6 - 8 + 2 = 0$ so angle θ_3 is a right angle as required, and hence the triangle does have a right angle.